Quadruple Covers of Algebraic Varieties

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Abstract

Let X and Y be varieties over a field k; $\pi : X \to Y$ is a quadruple cover of Y if $\pi_*\mathcal{O}_X$ is a locally free, rank 4 $\mathcal{O}_Y$-algebra. If char $k \neq 2$, we see that $\pi_*\mathcal{O}_X$ splits as $\mathcal{O}_Y \oplus \mathcal{E}$ where $\mathcal{E}$ is a locally free rank 3 sheaf over $\mathcal{O}_Y$, which is locally the 'trace zero' module. For each $y \in Y$, we therefore have a rank 4 associative, commutative algebra over $\mathcal{O}_{Y,y}$. We find that these algebras are parametrized by an affine cone over the Grassmanian $G(2,6)$ with vertex corresponding to the algebra $k[x,y,z]/(x,y,z)^2$. We then show that a quadruple cover with trace zero module $\mathcal{E}$ over a variety Y is determined by a totally decomposable section $\eta \in H^0(\wedge^2 S^2\ \mathcal{E}^* \otimes \wedge^3\ \mathcal{E})$. We then examine the case in which the section η has no zeros. Here, each rank 4 algebra may be associated to a pencil of conics. As a special case of this, we look at the work of G. Casnati on Gorenstein covers, and we show that his analysis is the subcase where the pencil of conics has length 4 base locus. Finally, we study the case in which the trace zero module $\mathcal{E}$ is split. In this context, Galois covers, which are covers induced by the action of a group of order 4 on the covering variety X, are also studied.

1 Introduction

In this article, we develop the theory of quadruple covers of algebraic varieties over a field of characteristic $\neq 2$. These are flat, finite maps of degree 4, i.e $\pi : X \to Y$ such that $\pi_*\mathcal{O}_X$ is a locally free rank 4 $\mathcal{O}_Y$-algebra. Geometrically, above every generic point $y \in Y$ there exist 4 points of X mapping to y, or more generally, the fiber over $y \in Y$ is a length four scheme. Given a variety Y, we seek the data on Y which is needed to build a quadruple cover. This is in the spirit of double covers, in which the building data is a line bundle L on Y and a divisor $D \in |2L|$. D is the branch locus of the cover, and the cover is built via $\mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L))$ where the ring structure is imposed by the embedding of $\mathcal{O}_Y(-2L) = \mathcal{O}_Y(-L) \otimes \mathcal{O}_Y(-L) \to \mathcal{O}_Y$. Alternatively, X is given locally by the equation $z^2 = f$ in the line bundle L on Y, where f is the local equation of D in $|2L|$ and z is the fiber coordinate. The reader may consult Persson [P1, P2] and Barth, Peters, and Van den Ven [B-P-V] for an exposition of this standard material in the case where X and Y are surfaces.

This is also in the spirit of triple covers as developed by Miranda [M], in the characteristic zero case, and Pardini [Pa] in positive characteristic. Our work will closely follow that of [M]. There the local question, "Which equations must be satisfied to build a rank three, associative, commutative algebra over a given ring?" is posed. Then a description of the relevant maps without reference to a choice of local basis is given, allowing the local analysis to *sheafify*. It is found that triple covers are determined by a locally free rank 2 $\mathcal{O}_Y$-module $\mathcal{E}$ and a section $\phi \in H^0(S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E})$.

In the case of quadruple covers, we first show that the locally free rank four $\mathcal{O}_Y$ algebra $\pi_*\mathcal{O}_X$ splits as $\mathcal{O}_Y \oplus \mathcal{E}$, where $\mathcal{E}$ is the locally free rank three $\mathcal{O}_Y$-module of 'trace - zero' elements. In the local analysis, we show that the equations which must be satisfied to build a rank four, associative, commutative algebra

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over $\mathcal{O}_Y$ are the equations of a $G(2,6)$, the Grassmanian parameterizing 2-dimensional subspaces of a six dimensional space. This is not surprising due to the work of Katz [K], who has shown that the deformations of fat points in $\mathbb{A}^3$ are parameterized by a cone over the Grassmanian $G(2,6)$ with a three dimensional vertex. The fat point is the variety determined by the ring $k[x,y,z]/(x^2, xy, xz, y^2, yz, z^2)$, and we may view this as the trivial algebra, with all others a deformation of the fat point. In our case, the additional trace-zero relations cut the vertex of the cone to a point.

We then describe the relevent maps without reference to a local basis. These maps arise from the following. To define an associative multiplication on $\pi_*\mathcal{O}_X$ is to give a map from $\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X$ to $\pi_*\mathcal{O}_X$. Since $\pi_*\mathcal{O}_X$ splits as $\mathcal{O}_Y \oplus \mathcal{E}$, using the left and right module structure of $\mathcal{E}$ over $\mathcal{O}_Y$, we restrict to finding a map from $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{E}$ to $\mathcal{O}_Y \oplus \mathcal{E}$. For the multiplication to be commutative, this map must factor through $S^2\mathcal{E}$. Therefore, the map we examine is the one, which we call ϕ, mapping $S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$. We decompose this map into $\phi_1 : S^2\mathcal{E} \to \mathcal{E}$ and $\phi_2 : S^2\mathcal{E} \to \mathcal{O}_Y$. It is shown in Theorem 5.4 that ϕ_2 is induced by the map ϕ_1. In Theorem 5.2 we prove the following:

Theorem 1.1. *There is a natural transformation from $\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$ to $Hom(S^2\mathcal{E}, \mathcal{E})$, such that a map ϕ_1 $\in Hom(S^2\mathcal{E}, \mathcal{E})$ induces an associative multiplication on $\mathcal{E}$ if and only if it is the image, under this natural transformation, of a totally decomposable section.*

Our local analysis then sheafifies, giving in Theorem 5.6 essentially the following:

Theorem 1.2. *A quadruple cover of Y is determined by a locally free rank 3 $\mathcal{O}_Y$-module $\mathcal{E}$ and a totally decomposable section $\eta \in H^0(\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E})$.*

We then examine two special cases of quadruple covers: first the one in which the fat point is avoided in every fiber of the cover, and second the one in which the trace zero module splits as a direct sum of line bundles. In the first case, we are able to associate to each algebra a pencil of conics. This is summarized in Table 6.2. A special case of such covers are Gorestein Covers, which have been studied by G. Casnati and T. Ekedahl [C-E], and we briefly put their work into our context. In the second case, we also examine Galois Covers, which are covers induced by the action of a group of order 4 on a variety. A special case of these is Bidouble Covers, which have been studied by F. Catanese [Ca], and we also put his work into our setting.

Double covers and, more recently, triple covers have been used to construct examples of varieties with certain desired properties. We believe that the study of quadruple covers will give us another tool in making these constructions. For the sake of brevity, long, exhaustive computations have been ommitted and replaced with examples of the work that was done. For complete details, see the first author's thesis [Ha].

2 The Trace Zero Module

Definition 2.1. Let X and Y be algebraic varieties over a field k, $\mathcal{O}_X$ and $\mathcal{O}_Y$ the sheaves of regular functions on X and Y respectively, and $\pi : X \to Y$ a regular map. We say that π is a *covering map of degree d* if $\pi_*\mathcal{O}_X$ is a locally free rank d $\mathcal{O}_Y$-algebra.

Locally, each $\alpha \in \pi_*\mathcal{O}_X$ defines a map $\pi_*\mathcal{O}_X \xrightarrow{\alpha} \pi_*\mathcal{O}_X$, given by multiplication by α. If we choose a local basis for $\pi_*\mathcal{O}_X$ over $\mathcal{O}_Y$, then multiplication by α determines a matrix over $\mathcal{O}_Y$ with respect to the chosen local basis. If char$(k) \neq$ d, we can then define a map $\frac{1}{d}$tr : $\pi_*\mathcal{O}_X \to \mathcal{O}_Y$ by $\alpha \to \frac{1}{d}$(trace of the matrix determined by α). This is independent of the choice of basis, since a change of basis simply conjugates the matrix of α, leaving the trace unchanged; hence $\dfrac{1}{d}$tr is globally defined.

Letting $\mathcal{E}$ be the kernel of the trace map, we have an exact sequence

$$0 \to \mathcal{E} \to \pi_*\mathcal{O}_X \xrightarrow{\frac{1}{d}tr} \mathcal{O}_Y \to 0.$$

If $\alpha \in \mathcal{O}_Y$, then the matrix determined by α is $\mathrm{diag}(\alpha, \alpha, ..., \alpha)$. Thus $\frac{1}{d}\mathrm{tr}(\alpha) = \alpha$ is the identity on $\mathcal{O}_Y$. Therefore, $\frac{1}{d}\mathrm{tr}\circ\pi^* : \mathcal{O}_Y \to \mathcal{O}_Y$ is the identity on $\mathcal{O}_Y$, thus the sequence splits, and we have proven the following:

Lemma 2.2. *$\pi_*\mathcal{O}_X$ splits as $\mathcal{O}_Y \oplus \mathcal{E}$, where $\mathcal{E}$ is the sub-module of 'trace-zero' elements.*

In the following we assume that $\mathrm{char}(k) \neq 2$, so that the above lemma holds for d=4. We seek information from a variety Y with which we may build a quadruple cover. The multiplication in $\pi_*\mathcal{O}_X$ is an $\mathcal{O}_Y$-linear map

$$\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_X$$

i.e.

$$[\mathcal{O}_Y \oplus \mathcal{E}] \otimes [\mathcal{O}_Y \oplus \mathcal{E}] \to \mathcal{O}_Y \oplus \mathcal{E}$$

or

$$(\mathcal{O}_Y \otimes \mathcal{O}_Y) \oplus (\mathcal{O}_Y \otimes \mathcal{E}) \oplus (\mathcal{E} \otimes \mathcal{O}_Y) \oplus (\mathcal{E} \otimes \mathcal{E}) \to \mathcal{O}_Y \oplus \mathcal{E}$$

The first three coordinates of this map are determined as the multiplication in $\mathcal{O}_Y$ and the left and right module structures of $\mathcal{E}$ over $\mathcal{O}_Y$. Thus we seek a map $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$.

For multiplication to be commutative, this map must factor through the second symmetric power of $\mathcal{E}$, i.e. we seek a map:

$$\phi : S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$$

Conversely, any map $\phi : S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$ will define a commutative operation in $\mathcal{O}_Y \oplus \mathcal{E}$.

3 Local Analysis

Let x, y, z be local generators for $\mathcal{E}$. A map $\phi : S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$ may then be specified by the image of $x^2, xy, xz, y^2, yz, z^2$ in $\mathcal{O}_Y \oplus \mathcal{E}$ and extending linearly, such that the induced multiplication in $\mathcal{O}_Y \oplus \mathcal{E}$ is associative. The trivial way to accomplish this is to take ϕ to be the zero map, which gives the 'fat point' algebra. We may then view all others as deformations of this, i.e. any ϕ may be thought of as a deformation of a fat point in $\mathbb{A}^3$ in which the trace zero relations hold. In light of this and the work of Katz [K], who has shown that the deformations of fat points in $\mathbb{A}^3$ are parameterized by a cone over the Grassmanian $G(2,6)$ with a three dimensional vertex, the following theorem is not surprising. The additional trace zero relations cut the vertex of the cone down to a single point representing the 'fat point' algebra.

Theorem 3.1. *Under a linear change of coordinates, the relations which must be satisfied for the multiplication in the trace-zero module $\mathcal{E}$ to be associative, along with the trace zero relations, are the equations of an affine cone over a G(2,6), the Grassmanian of the two dimensional subspaces of a six dimensional space, under its natural Plücker embedding in $\mathbb{P}^{14}$.*

Proof. Let x, y, z be local generators for the trace zero module $\mathcal{E}$, and write

$$x^2 = a_1x + a_2y + a_3z + b_1,$$

$$xy = a_4x + a_5y + a_6z + b_2,$$

$$xz = a_7x + a_8y + a_9z + b_3,$$

$$y^2 = a_{10}x + a_{11}y + a_{12}z + b_4,$$
$$yz = a_{13}x + a_{14}y + a_{15}z + b_5,$$
$$z^2 = a_{16}x + a_{17}y + a_{18}z + b_6.$$

For associativity, we require that the 8 equations

$$x^2y = x(xy), x^2z = x(xz), y^2x = y(xy), y^2z = y(yz), z^2x = z(xz), z^2y = z(yz), x(yz) = (xy)z = (xz)y,$$

hold, as well as the trace zero relatons

$$\begin{array}{rcccl} tr(x) & = & a_1 + a_5 + a_9 & = & 0 \\ tr(y) & = & a_4 + a_{11} + a_{15} & = & 0 \\ tr(z) & = & a_7 + a_{14} + a_{18} & = & 0. \end{array} \tag{3.2}$$

These 8 equations are quadratic in the structure constants $\{a_i, b_j\}$.

As an example of the computations, observe that

$$x^2y = a_1xy + a_2y^2 + a_3yz + b_1y$$
$$= a_1(a_4x + a_5y + a_6z + b_2) + a_2(a_{10}x + a_{11}y + a_{12}z + b_4) + a_3(a_{13}x + a_{14}y + a_{15}z + b_5) + b_1y$$
$$= (a_1a_4 + a_2a_{10} + a_3a_{13})x + (a_1a_5 + a_2a_{11} + a_3a_{14} + b_1)y$$
$$+(a_1a_6 + a_2a_{12} + a_3a_{15})z + (a_1b_2 + a_2b_4 + a_3b_5)$$

and

$$x(xy) = a_4x^2 + a_5xy + a_6xz + b_2x$$
$$= a_4(a_1x + a_2y + a_3z + b_1) + a_5(a_4x + a_5y + a_6z + b_2) + a_6(a_7x + a_8y + a_9z + b_3) + b_2x$$
$$= (a_1a_4 + a_4a_5 + a_6a_7 + b_2)x + (a_2a_4 + a_5^2 + a_6a_8)y$$
$$+(a_3a_4 + a_5a_6 + a_6a_9)z + (a_4b_1 + a_5b_2 + a_6b_3).$$

Equating coefficients yields the following:

$$a_1a_4 + a_2a_{10} + a_3a_{13} = a_1a_4 + a_4a_5 + a_6a_7 + b_2,$$

$$a_1a_5 + a_2a_{11} + a_3a_{14} + b_1 = a_2a_4 + a_5^2 + a_6a_8,$$

$$a_1a_6 + a_2a_{12} + a_3a_{15} = a_3a_4 + a_5a_6 + a_6a_9,$$

$$a_1b_2 + a_2b_4 + a_3b_5 = a_4b_1 + a_5b_2 + a_6b_3.$$

The 7 other equations yield 28 further quadratic equations, and the 32 total equations define the locus of associative multiplications.

Note that we may use the first and second equations to solve for b_2 and b_1 in terms of the a_i's, and thus reduce the number of variables. (Locally we are decomposing the map $\phi : S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$ into $\phi_1 : S^2\mathcal{E} \to \mathcal{E}$ and $\phi_2 : S^2\mathcal{E} \to \mathcal{O}_Y$.) The other equations give similar expressions for each b_i, and we have:

$$b_1 = -a_1a_5 + a_2a_4 - a_2a_{11} - a_3a_{14} + a_5^2 + a_6a_8$$
$$b_2 = a_2a_{10} + a_3a_{13} - a_4a_5 - a_6a_7$$
$$b_3 = a_2a_{13} + a_3a_{16} - a_4a_8 - a_7a_9$$

$$b_4 = -a_1a_{10} + a_4^2 - a_4a_{11} + a_5a_{10} + a_6a_{13} - a_7a_{12}$$
$$b_5 = -a_5a_{13} + a_8a_{10} + a_{12}a_{17} - a_{14}a_{15}$$
$$b_6 = -a_1a_{16} - a_4a_{17} + a_7^2 - a_7a_{18} + a_8a_{13} + a_9a_{16}.$$

Plugging these expressions for the b_i's into each of the 32 equations gives 6 identities, and 26 equations for the a_i's, some quadratic and some cubic. The cubic equations are easily seen to be in the ideal generated by the quadratic ones. It is proved in [Ha] that the quadratic equations generate the same ideal as the following 15 quadratic equations:

(a) $-a_6(-a_1 + a_5 + a_9) - a_3(a_4 - a_{15}) + a_2a_{12} = 0$

(b) $(a_5 - a_9)(-a_1 + a_5 + a_9) + a_2(a_4 - a_{11} + a_{15}) - a_3(a_7 + a_{14} - a_{18}) = 0$

(c) $a_8(-a_1 + a_5 + a_9) + a_2(a_7 - a_{14}) - a_3a_{17} = 0$

(d) $-a_6(a_4 - a_{11} + a_{15}) - a_{12}(a_5 - a_9) + a_3a_{10} = 0$

(e) $-a_6(a_7 - a_{14}) + a_3a_{13} - a_8a_{12} = 0$

(f) $(a_7 - a_{14})(a_5 - a_9) - a_8(a_4 - a_{11} + a_{15}) + a_3a_{16} = 0$

(g) $(a_4 - a_{15})(a_5 - a_9) + a_6(a_7 + a_{14} - a_{18}) - a_2a_{10} = 0$

(h) $a_8(a_4 - a_{15}) - a_2a_{13} + a_6a_{17} = 0$

(i) $a_8(a_7 + a_{14} - a_{18}) - a_{17}(a_5 - a_9) - a_2a_{16} = 0$

(j) $a_{13}(a_5 - a_9) - a_8a_{10} + a_6a_{16} = 0$

(k) $-(a_4 - a_{15})(a_4 - a_{11} + a_{15}) + a_{10}(a_1 - a_5 - a_9) + a_{12}(a_7 + a_{14} - a_{18}) = 0$

(l) $-(a_7 - a_{14})(a_4 - a_{15}) + a_{13}(a_1 - a_5 - a_9) + a_{12}a_{17} = 0$

(m) $-(a_7 - a_{14})(a_7 + a_{14} - a_{18}) - a_{16}(-a_1 + a_5 + a_9) + a_{17}(a_4 - a_{11} + a_{15}) = 0$

(n) $-a_{13}(a_4 - a_{11} + a_{15}) + a_{10}(a_7 - a_{14}) + a_{12}a_{16} = 0$

(o) $a_{13}(a_7 + a_{14} - a_{18}) - a_{16}(a_4 - a_{15}) - a_{10}a_{17} = 0$

Upon making the following change of coordinates,

$$\begin{array}{ll}
a_1 = \frac{1}{2}c_{2,3} & c_{1,2} = a_3 \\
a_2 = -c_{1,3} & c_{1,3} = -a_2 \\
a_3 = c_{1,2} & c_{1,4} = a_6 \\
a_4 = \frac{1}{4}c_{2,5} - \frac{1}{2}c_{3,4} & c_{1,5} = 2a_9 + a_1 \\
a_5 = -\frac{1}{2}c_{1,5} - \frac{1}{4}c_{2,3} & c_{1,6} = -a_8 \\
a_6 = c_{1,4} & c_{2,3} = 2a_1 \\
a_7 = \frac{1}{2}c_{2,6} - \frac{1}{4}c_{3,5} & c_{2,4} = a_{12} \\
a_8 = -c_{1,6} & c_{2,5} = -2a_{11} \\
a_9 = \frac{1}{2}c_{1,5} - \frac{1}{4}c_{2,3} & c_{2,6} = 2a_7 + a_{18} \\
a_{10} = c_{4,5} & c_{3,4} = 2a_{15} + a_{11} \\
a_{11} = -\frac{1}{2}c_{2,5} & c_{3,5} = 2a_{18} \\
a_{12} = c_{2,4} & c_{3,6} = -a_{17} \\
a_{13} = c_{4,6} & c_{4,5} = a_{10} \\
a_{14} = -\frac{1}{2}c_{2,6} - \frac{1}{4}c_{3,5} & c_{4,6} = a_{13} \\
a_{15} = \frac{1}{4}c_{2,5} + \frac{1}{2}c_{3,4} & c_{5,6} = a_{16} \\
a_{16} = c_{5,6} & \\
a_{17} = -c_{3,6} & \\
a_{18} = \frac{1}{2}c_{3,5} &
\end{array} \tag{3.3}$$

(keeping in mind the trace zero relations $a_1 + a_5 + a_9 = 0$, $a_4 + a_{11} + a_{15} = 0$, $a_7 + a_{14} + a_{18} = 0$, see 3.2), we now have:

$$
\begin{aligned}
c_{1,2}c_{3,4} - c_{1,3}c_{2,4} + c_{1,4}c_{2,3} &= 0\\
c_{1,2}c_{3,5} - c_{1,3}c_{2,5} + c_{1,5}c_{2,3} &= 0\\
c_{1,2}c_{3,6} - c_{1,3}c_{2,6} + c_{1,6}c_{2,3} &= 0\\
c_{1,2}c_{4,5} - c_{1,4}c_{2,5} + c_{1,5}c_{2,4} &= 0\\
c_{1,2}c_{4,6} - c_{1,4}c_{2,6} + c_{1,6}c_{2,4} &= 0\\
c_{1,2}c_{5,6} - c_{1,5}c_{2,6} + c_{1,6}c_{2,5} &= 0\\
c_{1,3}c_{4,5} - c_{1,4}c_{3,5} + c_{1,5}c_{3,4} &= 0\\
c_{1,3}c_{4,6} - c_{1,4}c_{3,6} + c_{1,6}c_{3,4} &= 0\\
c_{1,3}c_{5,6} - c_{1,5}c_{3,6} + c_{1,6}c_{3,5} &= 0\\
c_{1,4}c_{5,6} - c_{1,5}c_{4,6} + c_{1,6}c_{4,5} &= 0\\
c_{2,3}c_{4,5} - c_{2,4}c_{3,5} + c_{2,5}c_{3,4} &= 0\\
c_{2,3}c_{4,6} - c_{2,4}c_{3,6} + c_{2,6}c_{3,4} &= 0\\
c_{2,3}c_{5,6} - c_{2,5}c_{3,6} + c_{2,6}c_{3,5} &= 0\\
c_{2,4}c_{5,6} - c_{2,5}c_{4,6} + c_{2,6}c_{4,5} &= 0\\
c_{3,4}c_{5,6} - c_{3,5}c_{4,6} + c_{3,6}c_{4,5} &= 0.
\end{aligned}
\tag{3.4}
$$

These are the Plücker relations for a G(2,6) as long as not all of the coefficients are zero. When they are all zero, we have the fat point $k[x,y,z]/(x,y,z)^2$. This is the vertex of the cone. $\square$

For completeness, we list the constants b_i in terms of our new coordinates $c_{i,j}$'s :

$$
\begin{aligned}
b_1 &= \tfrac{1}{2}c_{1,2}c_{2,6} + \tfrac{1}{4}c_{1,2}c_{3,5} - \tfrac{3}{4}c_{1,3}c_{2,5} + \tfrac{1}{2}c_{1,3}c_{3,4} - c_{1,4}c_{1,6} + \tfrac{1}{4}c_{1,5}^2 + \tfrac{1}{2}c_{1,5}c_{2,3} + \tfrac{3}{16}c_{2,3}^2\\
b_2 &= c_{1,2}c_{4,6} - c_{1,3}c_{4,5} - \tfrac{1}{2}c_{1,4}c_{2,6} + \tfrac{1}{4}c_{1,4}c_{3,5} + \tfrac{1}{8}c_{1,5}c_{2,5} - \tfrac{1}{4}c_{1,5}c_{3,4} + \tfrac{1}{16}c_{2,3}c_{2,5} - \tfrac{1}{8}c_{2,3}c_{3,4}\\
b_3 &= c_{1,2}c_{5,6} - c_{1,3}c_{4,6} - \tfrac{1}{4}c_{1,5}c_{2,6} + \tfrac{1}{8}c_{1,5}c_{3,5} + \tfrac{1}{4}c_{1,6}c_{2,5} - \tfrac{1}{2}c_{1,6}c_{3,4} + \tfrac{1}{8}c_{2,3}c_{2,6} - \tfrac{1}{16}c_{2,3}c_{3,5}\\
b_4 &= c_{1,4}c_{4,6} - \tfrac{1}{2}c_{1,5}c_{4,5} - \tfrac{3}{4}c_{2,3}c_{4,5} - \tfrac{1}{2}c_{2,4}c_{2,6} + \tfrac{1}{4}c_{2,4}c_{3,5} + \tfrac{3}{16}c_{2,5}^2 - \tfrac{1}{2}c_{2,5}c_{3,4} + \tfrac{1}{4}c_{3,4}^2\\
b_5 &= c_{1,4}c_{5,6} - \tfrac{1}{2}c_{1,5}c_{4,6} - \tfrac{3}{4}c_{2,3}c_{4,5}\tfrac{1}{8}c_{2,5}c_{2,6} + \tfrac{1}{16}c_{2,5}c_{3,5} - \tfrac{3}{4}c_{2,6}c_{3,4} + \tfrac{1}{8}c_{3,4}c_{3,5}\\
b_6 &= \tfrac{1}{2}c_{1,5}c_{5,6} - c_{1,6}c_{4,6} - \tfrac{3}{4}c_{2,3}c_{5,6} + \tfrac{1}{4}c_{2,5}c_{3,6} + \tfrac{1}{4}c_{2,6}^2 - \tfrac{1}{2}c_{2,6}c_{3,5} - \tfrac{1}{2}c_{3,4}c_{3,6} + \tfrac{3}{16}c_{3,5}^2.
\end{aligned}
\tag{3.5}
$$

4 The Variety of Associative Algebras

In the last section, our discussion of the local analysis has been a discussion of associative, commutative, rank 4 algebras with identity over a given field. The variety of associative algebras of a given rank has been studied by several people over the years. In particular, the rank 4 case has been studied recently by P. Gabriel [G] and M. Schaps [S], just to name two. Gabriel has given a list of all rank 4 algebras over a field. He has shown that the commutative algebras form an irreducible component of the variety of associative algebras. Our work now shows what this variety is, an affine cone over a G(2,6) with vertex the fat point $k[x,y,z]/(x,y,z)^2$. Using what is known about G(2,6), we may state a few facts regarding this component.

Corollary 4.1. *Away from the fat point $k[x,y,z]/(x,y,z)^2$, the variety parametrizing associative, commutative algebras with identity is smooth. Also, the equations of associativity, commutativity, and identity generate the ideal of the variety.*

Proof. These are true for the quadratic relations (3.4) defining G(2,6). $\square$

5 Globalization

We begin with the map $\phi_1 : S^2\mathcal{E} \to \mathcal{E}$. It will be shown shortly that such a map will induce an associative multiplication on $\mathcal{E}$ if it is the image, under a natural map from $\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$ to $S^2\mathcal{E}^* \otimes \mathcal{E}$, of a totally decomposable element. We must first understand what we mean by a totally decomposable element in $\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$.

Let V be an n-dimensional vector space, and $F \subset V$ a k-dimensional subspace of V spanned by the vectors $v_1, v_2, ..., v_k$. We may associate to F the multivector $\lambda = v_1 \wedge ... \wedge v_k \in \wedge^k(V)$ (See Harris [H]). This is the Plücker embedding of $G(k, V)$ into $\mathbb{P}(\wedge^k V)$. A multivector $\omega \in \mathbb{P}(\wedge^k V)$ corresponds to a point on the Grassmannian $G(k, V)$ if it is a "pure wedge" or "totally decomposable" in $\mathbb{P}(\wedge^k V)$, i.e. if there exists vectors $v_1, ..., v_k$ such that $\omega = v_1 \wedge ... \wedge v_k$.

We may extend this idea to vector bundles $\mathcal{V}$ of rank n over Y, and k-dimensional subbundles $\mathcal{F}$ of $\mathcal{V}$ by doing so fiber by fiber. In particular, $\mathcal{E}$ is a locally free rank 3 sheaf on Y, and thus $S^2\mathcal{E}$ is a locally free rank 6 sheaf, i.e. a rank 6 vector bundle on Y (Hartshorne [Hr]). A section $\omega \,\epsilon\, \wedge^2 S^2\mathcal{E}$ is totally decomposable if, for each $y \in Y$, $(\omega)_y$ is totally decomposable in $(\wedge^2 S^2\mathcal{E})_y$. This leads us to the following:

Definition 5.1. A section $\psi \in \wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E} \cong \mathrm{Hom}(\wedge^3\mathcal{E}^*, \wedge^2 S^2\mathcal{E}^*)$ will be called *totally decomposable* if, for each $y \in Y$, the image of the restriction of ψ to each fiber, $\psi|_y : (\wedge^3\mathcal{E}^*)_y \to (\wedge^2 S^2\mathcal{E}^*)_y$, is a totally decomposable section of $(\wedge^2 S^2\mathcal{E}^*)_y$.

Before we state our theorem, let us define the natural map from $\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$ to $S^2\mathcal{E}^* \otimes \mathcal{E}$ which we will use. To define the natural transformation, note that $\mathrm{Hom}(S^2\mathcal{E}, \mathcal{E}) \cong S^2\mathcal{E}^* \otimes \mathcal{E}$ and we seek a natural element of $\mathrm{Hom}(\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}, S^2\mathcal{E}^* \otimes \mathcal{E}) \cong \wedge^2 S^2\mathcal{E} \otimes \wedge^3\mathcal{E}^* \otimes S^2\mathcal{E}^* \otimes \mathcal{E} \cong \mathrm{Hom}(S^2\mathcal{E} \otimes \wedge^3\mathcal{E}, \wedge^2 S^2\mathcal{E} \otimes \mathcal{E})$.

The natural element here is a map $A : S^2\mathcal{E} \otimes \wedge^3\mathcal{E} \to \wedge^2 S^2\mathcal{E} \otimes \mathcal{E}$ defined by the following formula on a typical generator:

$$ab \otimes c \wedge d \wedge e \to ac \wedge bd \otimes e - ad \wedge bc \otimes e + ad \wedge be \otimes c - ae \wedge bd \otimes c + ae \wedge bc \otimes d - ac \wedge be \otimes d.$$

Note that A is a well-defined, multilinear map (i.e. it is symmetric in a and b and skew-symmetric in c, d and e).

This map induces a map $B : \wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E} \to S^2\mathcal{E}^* \otimes \mathcal{E} \cong \mathrm{Hom}(S^2\mathcal{E}, \mathcal{E})$, given by

$$B(ab \wedge cd \otimes e \wedge f \wedge g)(hi) = (ab \wedge cd \otimes 1)(A(hi \otimes e \wedge f \wedge g)).$$

Theorem 5.2. *A map $\phi_1 \in Hom(S^2\mathcal{E}, \mathcal{E})$ induces an associative multiplication on $\mathcal{E}$ if and only if it is the image under the above natural transformation from $\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$ to $Hom(S^2\mathcal{E}, \mathcal{E})$ of a totally decomposable section.*

Proof. Let us examine the action of the maps A and B given above in terms of a local basis for $\mathcal{E}$. For the sake of brevity, we will demonstrate the computations needed. Let x, y, z be local generators for $\mathcal{E}$. Then natural choices for bases of the following modules are:

$$\wedge^3\mathcal{E} = \langle x \wedge y \wedge z \rangle,$$

$$S^2\mathcal{E} = \langle x^2, 2xy, 2xz, y^2, 2yz, z^2 \rangle,$$

$$S^2\mathcal{E}^* = \langle (x^2)^*, \frac{1}{2}x^*y^*, \frac{1}{2}x^*z^*, (y^2)^*, \frac{1}{2}y^*z^*, (z^2)^* \rangle.$$

Note that the given basis for $S^2\mathcal{E}^*$ is the dual of the basis for $S^2\mathcal{E}$.

The map A, then, sends

$$x^2 \otimes x \wedge y \wedge z \to x^2 \wedge xy \otimes z - xy \wedge x^2 \otimes z + xy \wedge xz \otimes x - xz \wedge xy \otimes x + xz \wedge x^2 \otimes y - x^2 \wedge xz \otimes y$$

$$= x^2 \wedge 2xy \otimes z + 2xy \wedge 2xz \otimes \frac{1}{2}x - x^2 \wedge 2xz \otimes y$$

To understand the induced map B, we may determine how $B(\alpha \otimes \beta)$ acts as an element of $\mathrm{Hom}(S^2\mathcal{E}, \mathcal{E})$ for $\alpha \otimes \beta$ in $\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$. Let γ be an element of $S^2\mathcal{E}$. Then $B(\alpha\otimes\beta)(\gamma) = (\alpha \otimes 1)(A(\gamma \otimes \beta)) \in \mathcal{O}_Y \otimes \mathcal{E} \cong \mathcal{E}$. In terms of the local bases, we have :

$$B((x^2)^* \wedge \frac{1}{2}x^*y^* \otimes x \wedge y \wedge z)(x^2)$$

$$= ((x^2)^* \wedge \frac{1}{2}x^*y^* \otimes 1)(A(x^2 \otimes x \wedge y \wedge z))$$

$$= ((x^2)^* \wedge \frac{1}{2}x^*y^* \otimes 1)(x^2 \wedge 2xy \otimes z + 2xy \wedge 2xz \otimes \frac{1}{2}x - x^2 \wedge 2xz \otimes y) = z$$

Continuing these computations, the action of $B(\alpha \otimes x \wedge y \wedge z)$ on $\gamma \in S^2\mathcal{E}$ may be summarized in Table 5.1.

Table 5.1: The Action of $B(\alpha \otimes x \wedge y \wedge z)$ on $\gamma \in S^2\mathcal{E}$

$\alpha\backslash\gamma$	x^2	xy	xz	y^2	yz	z^2
$(x^2)^* \wedge \frac{1}{2}x^*y^*$	z	0	0	0	0	0
$(x^2)^* \wedge \frac{1}{2}x^*z^*$	$-y$	0	0	0	0	0
$(x^2)^* \wedge (y^2)^*$	0	z	0	0	0	0
$(x^2)^* \wedge \frac{1}{2}y^*z^*$	0	$-\frac{1}{2}y$	$\frac{1}{2}z$	0	0	0
$(x^2)^* \wedge (z^2)^*$	0	0	$-y$	0	0	0
$\frac{1}{2}x^*y^* \wedge \frac{1}{2}x^*z^*$	$\frac{1}{2}x$	$-\frac{1}{4}y$	$-\frac{1}{4}z$	0	0	0
$\frac{1}{2}x^*y^* \wedge (y^2)^*$	0	0	0	z	0	0
$\frac{1}{2}x^*y^* \wedge \frac{1}{2}y^*z^*$	0	$\frac{1}{4}x$	0	$-\frac{1}{2}y$	$\frac{1}{4}z$	0
$\frac{1}{2}x^*y^* \wedge (z^2)^*$	0	0	$\frac{1}{2}x$	0	$-\frac{1}{2}y$	0
$\frac{1}{2}x^*z^* \wedge (y^2)^*$	0	$-\frac{1}{2}x$	0	0	$\frac{1}{2}z$	0
$\frac{1}{2}x^*z^* \wedge \frac{1}{2}y^*z^*$	0	0	$-\frac{1}{4}x$	0	$-\frac{1}{4}y$	$\frac{1}{2}z$
$\frac{1}{2}x^*z^* \wedge (y^2)^*$	0	0	0	0	0	$-y$
$(y^2)^* \wedge \frac{1}{2}y^*z^*$	0	0	0	x	0	0
$(y^2)^* \wedge (z^2)^*$	0	0	0	0	x	0
$\frac{1}{2}y^*z^* \wedge (z^2)^*$	0	0	0	0	0	x

Table 5.2: The Local Coefficients on the generators of $\wedge^2 S^2\mathcal{E}^*$

$c_{1,2}$	$(x^2)^* \wedge \frac{1}{2}x^*y^*$
$c_{1,3}$	$(x^2)^* \wedge \frac{1}{2}x^*z^*$
$c_{1,4}$	$(x^2)^* \wedge (y^2)^*$
$c_{1,5}$	$(x^2)^* \wedge \frac{1}{2}y^*z^*$
$c_{1,6}$	$(x^2)^* \wedge (z^2)^*$
$c_{2,3}$	$\frac{1}{2}x^*y^* \wedge \frac{1}{2}x^*z^*$
$c_{2,4}$	$\frac{1}{2}x^*y^* \wedge (y^2)^*$
$c_{2,5}$	$\frac{1}{2}x^*y^* \wedge \frac{1}{2}y^*z^*$
$c_{2,6}$	$\frac{1}{2}x^*y^* \wedge (z^2)^*$
$c_{3,4}$	$\frac{1}{2}x^*z^* \wedge (y^2)^*$
$c_{3,5}$	$\frac{1}{2}x^*z^* \wedge \frac{1}{2}y^*z^*$
$c_{3,6}$	$\frac{1}{2}x^*z^* \wedge (z^2)^*$
$c_{4,5}$	$(y^2)^* \wedge \frac{1}{2}y^*z^*$
$c_{4,6}$	$(y^2)^* \wedge (z^2)^*$
$c_{5,6}$	$\frac{1}{2}y^*z^* \wedge (z^2)^*$

If we associate the coefficients in Table 5.2 to the ordered basis of $\wedge^2 S^2\mathcal{E}^*$, i.e. a typical element in $\wedge^2 S^2\mathcal{E}^*$ is a sum $\sum_{i,j} c_{i,j} e_i \wedge e_j$, then the induced map in $\mathrm{Hom}(S^2\mathcal{E}, \mathcal{E})$ sends

$$x^2 \longrightarrow \frac{1}{2}c_{2,3}x - c_{1,3}y + c_{1,2}z$$

$$xy \longrightarrow (\frac{1}{4}c_{2,5} - \frac{1}{2}c_{3,4})x - (\frac{1}{2}c_{1,5} + \frac{1}{4}c_{2,3})y + c_{1,4}z$$

$$xz \longrightarrow (\frac{1}{2}c_{2,6} - \frac{1}{4}c_{3,5})x - c_{1,6}y + (\frac{1}{2}c_{1,5} - \frac{1}{4}c_{2,3})z$$

$$y^2 \longrightarrow c_{4,5}x - \frac{1}{2}c_{2,5}y + c_{2,4}z$$

$$yz \longrightarrow c_{4,6}x - (\frac{1}{2}c_{2,6} + \frac{1}{4}c_{3,5})y + (\frac{1}{2}c_{3,4} + \frac{1}{4}c_{2,5})z$$

$$z^2 \longrightarrow c_{5,6}x - c_{3,6}y + \frac{1}{2}c_{3,5}z.$$

From this, with our change of coordinates (3.3), we see that such a map will give an associative multiplication on $\mathcal{E}$ if and only if the $c_{i,j}$'s satisfy the Plücker relations. This is the criterion for ϕ_1 to be the image of a totally decomposable element.

□

Now, given $\phi_1 : S^2\mathcal{E} \to \mathcal{E}$, we next describe a natural way to determine ϕ_2 mapping $S^2\mathcal{E} \to \mathcal{O}_Y$ which gives the appropriate local constants b_j. The map ϕ_1 induces

$$\wedge^2\phi_1 : \wedge^2 S^2\mathcal{E} \to \wedge^2\mathcal{E},$$

or equivalently a map $\alpha : \wedge^2 S^2\mathcal{E} \otimes \wedge^2\mathcal{E}^* \to \mathcal{O}_Y$. We seek a natural element of $\mathrm{Hom}(S^2\mathcal{E}, \wedge^2 S^2\mathcal{E} \otimes \wedge^2\mathcal{E}^*)$ which we will compose with α. Now

$$\mathrm{Hom}(S^2\mathcal{E},\, \wedge^2 S^2\mathcal{E} \otimes \wedge^2\mathcal{E}^*) \cong\ S^2\mathcal{E}^* \otimes \wedge^2 S^2\mathcal{E} \otimes \wedge^2\mathcal{E}^* \cong \mathrm{Hom}(S^2\mathcal{E} \otimes \wedge^2\mathcal{E},\, \wedge^2 S^2\mathcal{E}).$$

The natural element in $\mathrm{Hom}(S^2\mathcal{E} \otimes \wedge^2\mathcal{E}, \wedge^2 S^2\mathcal{E})$, which we will call C, is defined on a typical generator $ab \otimes c \wedge d$ by

$$C(ab \otimes c \wedge d) \ = \frac{1}{2}(ad \wedge bc + bd \wedge ac)$$

and we extend linearly. This induces a map D in

$$\mathrm{Hom}(S^2\mathcal{E}, \wedge^2 S^2\mathcal{E} \otimes \wedge^2\mathcal{E}^*) \cong \mathrm{Hom}(S^2\mathcal{E},\, \mathrm{Hom}(\wedge^2\mathcal{E}, \wedge^2 S^2\mathcal{E}))$$

by the formula

$$D(ab)(c \wedge d) = C(ab \otimes c \wedge d).$$

Theorem 5.4. *The map $\phi_2 = \alpha \circ D$ in Hom$(S^2\mathcal{E},\, \mathcal{O}_Y)$, induced from the map ϕ_1 which gives an associative multiplication on $\mathcal{E}$, gives the appropriate local form of the constants.*

Proof. We will describe the maps C and D on local generators to see that the map ϕ_2 given by $\alpha \circ D$ gives the appropriate constants. Again, for the sake of brevity, we will illustrate just one of the computations needed. Let x, y, z be local generators for $\mathcal{E}$. We will show where ϕ_2 sends x^2.

First, the map C sends

$$\begin{array}{rcl} x^2 \otimes x \wedge y & \to & -x^2 \wedge xy \\ x^2 \otimes x \wedge z & \to & -x^2 \wedge xz \\ x^2 \otimes y \wedge z & \to & -xy \wedge xz \end{array} \tag{5.5}$$

To understand the induced map D, we show how $D(ab)$ acts as an element of $\mathrm{Hom}(\wedge^2\mathcal{E},\, \wedge^2 S^2\mathcal{E})$ for $ab \in S^2\mathcal{E}$. For example,

$$D(x^2)(x \wedge y) = C(x^2 \otimes x \wedge y) = -x^2 \wedge xy.$$

Continuing these computations, the action of $D(ab)$ on $\gamma \in \wedge^2\mathcal{E}$ for $ab \in S^2\mathcal{E}$ may be summarized in Table 5.3

Table 5.3: The Action of $D(ab)$ on $\gamma \in \wedge^2\mathcal{E}$

$ab \backslash \gamma$	$x \wedge y$	$x \wedge z$	$y \wedge z$
x^2	$-x^2 \wedge xy$	$-x^2 \wedge xz$	$-xy \wedge xz$
xy	$-\frac{1}{2}x^2 \wedge y^2$	$-\frac{1}{2}(xy \wedge xz + x^2 \wedge yz)$	$\frac{1}{2}(xz \wedge y^2 - xy \wedge z^2)$
xz	$\frac{1}{2}(xy \wedge xz - x^2 \wedge yz)$	$-\frac{1}{2}x^2 \wedge z^2$	$\frac{1}{2}(xz \wedge yz - xy \wedge z^2)$
y^2	$-xy \wedge y^2$	$-xy \wedge yz$	$-y^2 \wedge yz$
yz	$-\frac{1}{2}(xz \wedge y^2 + xy \wedge yz)$	$-\frac{1}{2}(xz \wedge yz + xy \wedge z^2)$	$-\frac{1}{2}y^2 \wedge z^2$
z^2	$-xz \wedge yz$	$-xz \wedge z^2$	$-yz \wedge z^2$

Therefore, $D(x^2) = -x^2 \wedge xy \otimes x^* \wedge y^* - x^2 \wedge xz \otimes x^* \wedge z^* - xy \wedge xz \otimes y^* \wedge z^*$, and to see where ϕ_2 sends x^2, we must know where α sends each of these terms. For example, to find $\alpha(x^2 \wedge xy \otimes x^* \wedge y^*)$, we observe that the map $\wedge^2\phi_1$ sends $x^2 \wedge xy \to (a_1a_5 - a_2a_4)x \wedge y + (a_1a_6 - a_3a_4)x \wedge z + (a_2a_6 - a_3a_5)y \wedge z$. Thus $\alpha(x^2 \wedge xy \otimes x^* \wedge y^*) = a_1a_5 - a_2a_4$. Under our change of coordinates, we have

$$\alpha(x^2 \wedge xy \otimes x^* \wedge y^*) = \frac{1}{4}\, c_{2,3}\, c_{1,5} - \frac{1}{8}\, {c_{2,3}}^2 + \frac{1}{4}\, c_{1,3}\, c_{2,5} - \frac{1}{2}\, c_{1,3}\, c_{3,4}.$$

Continuing these computations for α, we see that

$$\alpha(x^2 \wedge xz \otimes x^* \wedge z^*) = \frac{1}{4}\, c_{2,3}\, c_{1,5} - \frac{1}{8}\, {c_{2,3}}^2 - \frac{1}{2}\, c_{1,2}\, c_{2,6} + \frac{1}{4}\, c_{1,2}\, c_{3,5},$$

and

$$\alpha(xy \wedge xz \otimes y^* \wedge z^*) = -\frac{1}{4}\, {c_{1,5}}^2 + \frac{1}{16}\, {c_{2,3}}^2 + c_{1,4}\, c_{1,6}.$$

Finally,

$$\phi_2(x^2) = \frac{1}{2}c_{1,2}c_{2,6} - \frac{1}{4}c_{1,2}c_{3,5} - \frac{1}{4}c_{1,3}c_{2,5} + \frac{1}{2}c_{1,3}c_{3,4} - c_{1,4}c_{1,6} + \frac{1}{4}c_{1,5}^2 + \frac{3}{16}c_{2,3}^2.$$

In this way, we compute the image of each generator of $S^2\mathcal{E}$ under ϕ_2:

$$\phi_2(x^2) = \frac{1}{2}c_{1,2}c_{2,6} - \frac{1}{4}c_{1,2}c_{3,5} - \frac{1}{4}c_{1,3}c_{2,5} + \frac{1}{2}c_{1,3}c_{3,4} - c_{1,4}c_{1,6} + \frac{1}{4}c_{1,5}^2 + \frac{3}{16}c_{2,3}^2$$

$$\phi_2(xy) = \frac{1}{2}c_{1,2}c_{4,6} - \frac{1}{2}c_{1,3}c_{4,5} - \frac{1}{4}c_{1,4}c_{3,5} + \frac{1}{8}c_{1,5}c_{2,5} + \frac{1}{4}c_{1,5}c_{3,4} - \frac{1}{2}c_{1,6}c_{2,4} + \frac{1}{16}c_{2,3}c_{2,5} - \frac{1}{8}c_{2,3}c_{3,4}$$

$$\phi_2(xz) = \frac{1}{2}c_{1,2}c_{5,6} - \frac{1}{2}c_{1,3}c_{4,6} - \frac{1}{2}c_{1,4}c_{3,6} + \frac{1}{4}c_{1,5}c_{2,6} + \frac{1}{8}c_{1,5}c_{3,5} - \frac{1}{4}c_{1,6}c_{2,5} + \frac{1}{8}c_{2,3}c_{2,6} - \frac{1}{16}c_{2,3}c_{3,5}$$

$$\phi_2(y^2) = c_{1,4}c_{4,6} - \frac{1}{2}c_{1,5}c_{4,5} - \frac{1}{4}c_{2,3}c_{4,5} - \frac{1}{2}c_{2,4}c_{2,6} - \frac{1}{4}c_{2,4}c_{3,5} + \frac{3}{16}c_{2,5}^2 + \frac{1}{4}c_{3,4}^2$$

$$\phi_2(yz) = \frac{1}{2}c_{1,4}c_{5,6} - \frac{1}{2}c_{1,6}c_{4,5} - \frac{1}{4}c_{2,3}c_{4,6} - \frac{1}{2}c_{2,4}c_{3,6} + \frac{1}{8}c_{2,5}c_{2,6} + \frac{1}{16}c_{2,5}c_{3,5} - \frac{1}{4}c_{2,6}c_{3,4} + \frac{1}{8}c_{3,4}c_{3,5}$$

$$\phi_2(z^2) = \frac{1}{2}c_{1,5}c_{5,6} - c_{1,6}c_{4,6} - \frac{1}{4}c_{2,3}c_{5,6} - \frac{1}{4}c_{2,5}c_{3,6} + \frac{1}{4}c_{2,6}^2 - \frac{1}{2}c_{3,4}c_{3,6} + \frac{3}{16}c_{3,5}^2.$$

Recalling the equations for the constants in terms of the Plücker coordinates $c_{i,j}$'s (see 3.5), we have

$$\phi_2(x^2) - b_1 = -\frac{1}{2}(c_{1,2}c_{3,5} - c_{1,3}c_{2,5} + c_{1,5}c_{2,3})$$

$$\phi_2(xy) - b_2 = -\frac{1}{2}(c_{1,2}c_{4,6} - c_{1,4}c_{2,6} + c_{1,6}c_{2,4}) + \frac{1}{2}(c_{1,3}c_{4,5} - c_{1,4}c_{3,5} + c_{1,5}c_{3,4})$$

$$\phi_2(xz) - b_3 = -\frac{1}{2}(c_{1,2}c_{5,6} - c_{1,5}c_{2,6} + c_{1,6}c_{2,5}) + \frac{1}{2}(c_{1,3}c_{4,5} - c_{1,4}c_{3,5} + c_{1,5}c_{3,4})$$

$$\phi_2(y^2) - b_4 = \frac{1}{2}(c_{2,3}c_{4,5} - c_{2,4}c_{3,5} + c_{2,5}c_{3,4})$$

$$\phi_2(yz) - b_5 = -\frac{1}{2}(c_{1,4}c_{5,6} - c_{1,5}c_{4,6} + c_{1,6}c_{4,5}) + \frac{1}{2}(c_{2,3}c_{4,6} - c_{2,4}c_{3,6} + c_{2,6}c_{3,4})$$

$$\phi_2(z^2) - b_6 = \frac{1}{2}(c_{2,3}c_{5,6} - c_{2,5}c_{3,6} + c_{2,6}c_{3,5}).$$

Since ϕ_1 induces an associative multiplication on $\mathcal{E}$, the Plücker relations hold. Therefore, the right hand sides of the above equations are all zero, and ϕ_2 gives the appropriate local form of the constants.

□

Since the relevant maps have now been described without reference to a choice of basis, the local analysis 'sheafifies' to give us the following theorem.

Theorem 5.6. *Given a variety Y, a locally free rank 3 sheaf $\mathcal{E}$ on Y, and a totally decomposable section $\eta \in H^0(\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E})$, let*

$$\phi_1 = B(\eta) \,:\, S^2\mathcal{E} \to \mathcal{E},$$

and

$$\phi_2 = \alpha \circ D \,:\, S^2\mathcal{E} \to \mathcal{O}_Y,$$

where B and D are the natural transformations of Theorem 5.2 and Theorem 5.4 respectively, and α is the map induced by $\wedge^2\phi_1$. Then

$$\phi \;:=\; \phi_1 \oplus \phi_2$$

gives an associative multiplication on $\mathcal{O}_Y \oplus \mathcal{E}$, and $\pi : X \to Y$ is a quadruple cover of Y, where $X := \underline{Spec}(\mathcal{O}_Y \oplus \mathcal{E})$ and π is the map induced by the inclusion $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y \oplus \mathcal{E}$.

Conversely, if $\pi : X \to Y$ is a quadruple cover of Y, then $\pi_*(\mathcal{O}_X)$ splits as $\mathcal{O}_Y \oplus \mathcal{E}$, where $\mathcal{E}$ is the trace zero module. Furthermore, the multiplication in $\mathcal{O}_Y \oplus \mathcal{E}$ gives a totally decomposable section of $H^0(\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E})$.

6 Fat Point Avoidance and The Gorenstein Case

6.1 Fat Point Avoidance

In the previous chapter, we have seen that to build a quadruple cover over a variety Y, we need a locally free rank 3 sheaf $\mathcal{E}$ on Y and a totally decomposable section $\eta \in H^0(\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E})$. One way to obtain such a section is the following. First, given $\mathcal{E}$, take a rank 2 subbundle $\mathcal{F}$ of $S^2\mathcal{E}^*$ such that $\wedge^2\mathcal{F} \cong \wedge^3\mathcal{E}^*$. Then the Plücker embedding $p : \wedge^2\mathcal{F} \to \wedge^2 S^2\mathcal{E}^*$ gives a map $\tilde{p} : \wedge^3\mathcal{E}^* \to \wedge^2 S^2\mathcal{E}^*$, i.e. a totally decomposable section η of $H^0(\wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E})$.

In this case, since $\mathcal{F}$ is a subbundle of $S^2\mathcal{E}^*$, the section η will have no zeros in any fiber. Since η having zero in a fiber corresponds to the cover having the fat point $k[x, y, z]/(x, y, z)^2$ as its algebra in that fiber, covers built in this way will avoid the fat point in their fibers.

Let us consider this rank two subbundle $\mathcal{F}$. If $\mathcal{E}$ is generated locally by x, y, z, then $S^2\mathcal{E}$ is generated by the quadrics in these variables. Thus, to choose a rank 2 subbundle of $S^2\mathcal{E}^*$ is to locally choose a pair of conics in the dual variables x^*, y^*, z^*. Thus, to every associative, commutative rank 4 algebra over k which is not $k[x, y, z]/(x, y, z)^2$, there corresponds a pencil of conics. P. Gabriel in [G] has listed these algebras (see Table 6.1), and we now determine the pencil of conics corresponding to each. Again, we will demonstrate the computations on two examples.

Case 1 $\quad k \oplus k \oplus k \oplus k$

Let us examine $k \oplus k \oplus k \oplus k$. Let $e_0 = (1, 1, 1, 1)$ be the identity element. Generators for the trace zero module E over k are

$$e_1 = (-1, 1, 0, 0),\; e_2 = (-1, 0, 1, 0),\; e_3 = (-1, 0, 0, 1).$$

Note that

$$e_1^2 = (1, 1, 0, 0) = \frac{1}{2}(e_0 + e_1 - e_2 - e_3)$$

$$e_1e_2 = (1, 0, 0, 0) = \frac{1}{4}(e_0 - e_1 - e_2 - e_3)$$

$$e_1e_3 = (1, 0, 0, 0) = \frac{1}{4}(e_0 - e_1 - e_2 - e_3)$$

$$e_2^2 = (1, 0, 1, 0) = \frac{1}{2}(e_0 - e_1 + e_2 - e_3)$$

$$e_2e_3 = (1, 0, 0, 0) = \frac{1}{4}(e_0 - e_1 - e_2 - e_3)$$

Table 6.1: Commutative, Associative Algebras of Rank 4 over k

1.	$k \oplus k \oplus k \oplus k$
2.	$k \oplus k \oplus k[x]/(x^2)$
3.	$k[x]/(x^2) \oplus k[x]/(x^2)$
4.	$k \oplus k[x]/(x^3)$
5.	$k \oplus k[x,y]/(x,y)^2$
6.	$k[x]/(x^4)$
7.	$k[x,y]/(x^2,y^2)$
8.	$k[x,y]/(x^3,xy,y^2)$
9.	$k[x,y,z]/(x,y,z)^2$

$$e_3^2 = (1,0,0,1) = \frac{1}{2}(e_0 - e_1 - e_2 + e_3)$$

and

$$tr(e_1) = 0$$
$$tr(e_2) = 0$$
$$tr(e_3) = 0.$$

The structure coefficients for this algebra are then

$$a_1 = a_{11} = a_{18} = \frac{1}{2},$$

$$a_2 = a_3 = a_{10} = a_{12} = a_{16} = a_{17} = -\frac{1}{2},$$

$$a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{13} = a_{14} = a_{15} = -\frac{1}{4},$$

and

$$b_1 = b_4 = b_6 = \frac{1}{2}, b_2 = b_3 = b_5 = \frac{1}{4}.$$

Under our change of coordinates (3.3), the Plücker coefficients are

$$c_{1,2} = -\frac{1}{2},\ c_{1,3} = \frac{1}{2},\ c_{1,4} = -\frac{1}{4},\ c_{1,5} = 0,\ c_{1,6} = \frac{1}{4},$$

$$c_{2,3} = 1,\ c_{2,4} = -\frac{1}{2},\ c_{2,5} = -1,\ c_{2,6} = 0,$$

$$c_{3,4} = 0,\ c_{3,5} = 1,\ c_{3,6} = \frac{1}{2},$$

$$c_{4,5} = -\frac{1}{2},\ c_{4,6} = -\frac{1}{4},\ c_{5,6} = -\frac{1}{2}.$$

These arise as the two by two minors of the following matrix:

$$\begin{pmatrix} 0 & -1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -1 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

Associating to each column of this matrix the basis elements

$$(x^2)^*, \frac{1}{2}x^*y^*, \frac{1}{2}x^*z^*, (y^2)^*, \frac{1}{2}y^*z^*, (z^2)^*,$$

respectively, we see that two generators of the pencil are

$$-\frac{1}{2}x^*y^* + \frac{1}{2}x^*z^* - \frac{1}{2}(y^2)^* + \frac{1}{2}(z^2)^* = 0$$

and

$$-\frac{1}{2}(x^2)^* - \frac{1}{2}x^*y^* + \frac{1}{2}y^*z^* + \frac{1}{2}(z^2)^* = 0.$$

The corresponding 3 by 3 symmetric matrices for these conics, after multiplying both by 2 are

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

These give the pencil of conics determined by the Segre symbol $[1,1,1]$, and every member of this pencil meets in 4 distinct points.

Case 2 $\quad k \oplus k \oplus k[x]/(x^2)$

Let $e_0 = (1,1,1)$ be the identity element. Generators for the trace zero module E over k are

$$e_1 = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}),\ e_2 = (0,0,x),\ e_3 = (-\frac{1}{2}, \frac{1}{2}, x).$$

Note that

$$e_1^2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = \frac{1}{4}e_0$$

$$e_1e_2 = (0,0,\frac{1}{2}x) = \frac{1}{2}e_2$$

$$e_1e_3 = (\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}x) = e_2 - \frac{1}{2}e_3$$

$$e_2^2 = 0$$

$$e_2e_3 = 0$$

$$e_3^2 = (\frac{1}{4}, \frac{1}{4}, 0) = \frac{1}{8}e_0 - \frac{1}{4}e_1$$

and

$$tr(e_1) = tr(e_2) = tr(e_3) = 0.$$

The structure coefficients of this algebra are then

$$a_5 = \frac{1}{2}, a_8 = 1, a_9 = -\frac{1}{2}, a_{16} = -\frac{1}{4}, b_1 = \frac{1}{4}, b_6 = \frac{1}{8},$$

and all others zero. The corresponding Plücker coordinates are

$$c_{1,5} = -1, c_{1,6} = -1, c_{5,6} = -\frac{1}{4},$$

and all others zero. These arise as the two by two minors of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

Here, two generators of this pencil are the conics

$$(x^2)^* + \frac{1}{2}(\frac{1}{2}y^*z^*) + \frac{1}{4}(z^2)^* = 0$$

and

$$-\frac{1}{2}y^*z^* - (z^2)^* = 0.$$

This is the pencil of conics determined by the Segre symbol [2, 1].

We summarize our work in Table 6.2.

6.2 Gorenstein Covers

The case of Gorenstein Covers has recently been studied by G. Casnati and T. Ekedahl in [C-E]. They state that a cover $\pi : X \to Y$ is Gorenstein if the scheme-theoretic fiber $X_y := \pi^{-1}(y)$ is Gorenstein for every $y \in Y$. In this case, given the exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_*\mathcal{O}_X \to \mathcal{E} \to 0,$$

the Gorenstein condition is equivalent to the existence of an invertible relative dualizing sheaf $\omega_{X|Y}$ which, with relative duality, gives an exact sequence

$$0 \to \mathcal{E}^* \to \pi_*\omega_{X|Y} \to \mathcal{O}_Y \to 0.$$

Let $\mathbb{P}(\mathcal{E}) := \mathbf{Proj}(S\mathcal{E})$ denote the projective bundle on Y, with $\rho : \mathbb{P}(\mathcal{E}) \to Y$ the map induced by the natural monomorphism $\mathcal{O}_Y \to S\mathcal{E}$. Also, if $\phi : \mathcal{F} \to \mathcal{G}$ is a map between locally free $\mathcal{O}_Y$ sheaves, let $D_r(\phi) := \{y \in Y \mid rk\ \phi_y \leq r\}$. Casnati and Ekedahl show that if $\pi : X \to Y$ is a Gorenstein cover of degree 4, then there exists a unique $\mathbb{P}^2$-bundle $\rho : \mathbb{P} \to Y$ and an embedding $i : X \hookrightarrow \mathbb{P}$ such that $\pi = \rho \circ i$ with $\mathbb{P} \cong \mathbb{P}(\mathcal{E}^*)$ where $\mathcal{E} \cong$ coker $\pi^\sharp$.

We have the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[d, "\pi"] \\ \mathbb{P} \arrow[r, "\rho"] & Y \end{tikzcd}$$

Table 6.2: Rank 4 Non-Fat Pont Algebras and Their Associated Pencils

Algebra	Pencil Generators	Segre Symbol	Gorenstein?
$k \oplus k \oplus k \oplus k$	$-\frac{1}{2}x^*y^* + \frac{1}{2}x^*z^* - \frac{1}{2}(y^2)^* + \frac{1}{2}(z^2)^*$ $-\frac{1}{2}(x^2)^* - \frac{1}{2}x^*y^* + \frac{1}{2}y^*z^* + \frac{1}{2}(z^2)^*$	$[1,1,1]$	Yes
$k \oplus k \oplus k[x]/(x^2)$	$(x^2)^* + \frac{1}{2}(\frac{1}{2}y^*z^*) + \frac{1}{4}(z^2)^*$ $-\frac{1}{2}y^*z^* - (z^2)^*$	$[2,1]$	Yes
$k[x]/(x^2) \oplus k[x]/(x^2)$	$(z^2)^*$ $(x^2)^* - (y^2)^* + 2(\frac{1}{2}y^*z^*)$	$[(1,1),1]$	Yes
$k \oplus k[x]/(x^3)$	$\frac{1}{2}y^*z^*$ $2(\frac{1}{2}x^*z^*) + (y^2)^*$	$[3]$	Yes
$k \oplus k[x,y]/(x,y)^2$	$\frac{1}{2}x^*z^*$ $\frac{1}{2}y^*z^*$	Singular Pencil	No
$k[x]/(x^4)$	$(y^2)^*$ $2(\frac{1}{2}x^*y^*) + (z^2)^*$	$[(2,1)]$	Yes
$k[x,y]/(x^2,y^2)$	$(x^2)^*$ $(y^2)^*$	Singular Pencil	Yes
$k[x,y]/(x^3,xy,y^2)$	$(x^2)^*$ $-\frac{1}{2}x^*z^*$	Singular Pencil	No

In addition, there is an exact sequence of locally free $\mathcal{O}_{\mathbb{P}}$ sheaves

$$0 \to \rho^*(\wedge^3\mathcal{E}^*(-4)) \to \mathcal{N}(-2) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0$$

with $\mathcal{N} \cong \rho^*\mathcal{F}$ for some locally free $\mathcal{O}_Y$ sheaf $\mathcal{F}$. The natural isomorphism

$$\Phi : H^0(Y, S^2\mathcal{E}^* \otimes \mathcal{F}^*) \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{N}^*(2))$$

yields a section $\eta := \Phi^{-1}(\delta) \in H^0(Y, \mathcal{F}^* \otimes S^2\mathcal{E}^*)$.

Definition 6.1. Let $Y, \mathcal{E}, \mathcal{F}, \delta$, and η be as above. We say that η *has the right codimension at* $y \in Y$ if $\dim(D_0(\delta_y)) = 0$.

Casnati and Ekedahl then prove the following theorem.

Theorem 6.2. *Let Y be an integral scheme. Any Gorenstein cover $\pi : X \to Y$ of degree 4 with $\mathcal{E} \cong \operatorname{coker} \pi^\sharp$ determines a locally free $\mathcal{O}_Y$ sheaf $\mathcal{F}$ of rank 2 with $\wedge^2\mathcal{F} \cong \wedge^3\mathcal{E}^*$ and, up to scalars, a section $\eta \in H^0(Y, \mathcal{F}^* \otimes S^2\mathcal{E}^*)$ having the right codimension at every $y \in Y$.*

Conversely, given a locally free $\mathcal{O}_Y$ sheaf $\mathcal{F}$ of rank 2 with $\wedge^2\mathcal{F} \cong \wedge^3\mathcal{E}^*$ and $\eta \in H^0(Y, \mathcal{F}^* \otimes S^2\mathcal{E}^*)$ having the right codimension at every $y \in Y$, let $X := D_0(\Phi(\eta)) \subseteq \mathbb{P}$. Then the restriction π to X of the canonical map $\rho : \mathbb{P} \to Y$ is a Gorenstein cover of degree 4 such that $\mathcal{E} \cong coker\ \pi^\sharp$ and $\mathcal{F} \cong ker(S^2\mathcal{E}^* \to \pi_*\omega^2_{X|Y})$.

The section $\eta \in H^0(Y, \mathcal{F}^* \otimes S^2\mathcal{E}^*)$ gives a map $\eta : \mathcal{F} \to S^2\mathcal{E}^*$. We then have a map $\wedge^2\eta : \wedge^2\mathcal{F} \to \wedge^2 S^2\mathcal{E}^*$ or, since $\wedge^2\mathcal{F} \cong \wedge^3\mathcal{E}^*$, a map $\wedge^2\eta : \wedge^3\mathcal{E}^* \to \wedge^2 S^2\mathcal{E}^*$. This is the type of map we have seen which avoids the fat point. In addition, the restriction that η have the right codimension at every $y \in Y$ is the condition that, locally, the generators of the pencil of conics associated to each local algebra do not share a common component. In Table 6.2, these are the pencils which have an associated Segre symbol, as well as the algebra $k[x,y]/(x^2,y^2)$. We see immediately that the corresponding algebras are Gorenstein. In our case, the degree four covering X is isomorphic to the family of fiber-by-fiber base loci of the pencils of conics.

7 The Split Case and Galois Covers

7.1 The Split Case

We now analyze the case where $\mathcal{E}$ is split. Let $\mathcal{E} \cong L^{-1} \oplus M^{-1} \oplus N^{-1}$, where L, M, N are line bundles on Y. Then

$$\wedge^3\mathcal{E} \cong L^{-1} \otimes M^{-1} \otimes N^{-1}$$

$$S^2\mathcal{E} \cong L^{-2} \oplus L^{-1}M^{-1} \oplus L^{-1}N^{-1} \oplus M^{-2} \oplus M^{-1}N^{-1} \oplus N^{-2},$$

$$S^2\mathcal{E}^* \cong L^2 \oplus LM \oplus LN \oplus M^2 \oplus MN \oplus N^2,$$

$$\begin{aligned} \wedge^2 S^2\mathcal{E}^* \cong\ & (L^3 \otimes M) \oplus (L^3 \otimes N) \oplus (L^2 \otimes M^2) \oplus (L^2 \otimes M \otimes N) \oplus (L^2 \otimes N^2) \\ & \oplus (L^2 \otimes M \otimes N) \oplus (L \otimes M^3) \oplus (L \otimes M^2 \otimes N) \oplus (L \otimes M \otimes N^2) \oplus (L \otimes M^2 \otimes N) \\ & \oplus (L \otimes M \otimes N^2) \oplus (L \otimes N^3) \oplus (M^3 \otimes N) \oplus (M^2 \otimes N^2) \oplus (M \otimes N^3), \end{aligned}$$

and

$$\begin{aligned} \wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E} \cong\ & (L^2 \otimes N^{-1}) \oplus (L^2 \otimes M^{-1}) \oplus (L \otimes M \otimes N^{-1}) \oplus L \oplus (L \otimes M^{-1} \otimes N) \oplus L \\ & \oplus (M^2 \otimes N^{-1}) \oplus M \oplus N \oplus M \oplus N \oplus (M^{-1} \otimes N^2) \\ & \oplus (L^{-1} \otimes M^2) \oplus (L^{-1} \otimes M \otimes N) \oplus (L^{-1} \otimes N^2). \end{aligned}$$

If we now associate local generators x, y, z to L^{-1}, M^{-1}, N^{-1} respectively and local coordinates $c_{i,j}$'s as in Table 5.2, we see then that

$$\begin{aligned}
c_{1,2} &\in H^0(L^2 \otimes N^{-1}) \\
c_{1,3} &\in H^0(L^2 \otimes M^{-1}) \\
c_{1,4} &\in H^0(L \otimes M \otimes N^{-1}) \\
c_{1,5} &\in H^0(L) \\
c_{1,6} &\in H^0(L \otimes M^{-1} \otimes N) \\
c_{2,3} &\in H^0(L) \\
c_{2,4} &\in H^0(M^2 \otimes N^{-1}) \\
c_{2,5} &\in H^0(M) \\
c_{2,6} &\in H^0(N) \\
c_{3,4} &\in H^0(M) \\
c_{3,5} &\in H^0(N) \\
c_{3,6} &\in H^0(M^{-1} \otimes N^2) \\
c_{4,5} &\in H^0(L^{-1} \otimes M^2) \\
c_{4,6} &\in H^0(L^{-1} \otimes M \otimes N) \\
c_{5,6} &\in H^0(L^{-1} \otimes N^2).
\end{aligned}$$

Note that the Plücker relations occur in appropriate sheaves, e.g. the relation

$$c_{1,2}c_{3,4} - c_{1,3}c_{2,4} + c_{1,4}c_{2,3} = 0$$

is a relation in $H^0(L^2 \otimes M \otimes N^{-1})$.

7.2 Galois Covers

Definition 7.1. Let X be an algebraic variety, and G be a group of order 4. The action of G on X partitions X into equivalence classes. The space $Y = X \backslash G$ is also a variety, and the map $\pi : X \to Y$ is a quadruple cover of Y. We will call such a cover a *Galois quadruple cover.*

The action of G splits $\pi_* \mathcal{O}_X$ into a direct sum of eigenspaces. Thus, Galois covers are special cases of the trace zero module $\mathcal{E}$ being split. Up to isomorphism, there are two groups G of order 4. We will examine each of these groups individually.

Case 1 $G \cong \mathbb{Z}/4\mathbb{Z}$

Let G be the cyclic group of order 4 (i.e. $G =< \sigma | \sigma^4 = identity >$). Locally, let

$$E_i = \{f \in \mathcal{O}_X | \sigma(f) = if\}$$

$$E_{-1} = \{f \in \mathcal{O}_X | \sigma(f) = -f\}$$

$$E_{-i} = \{f \in \mathcal{O}_X | \sigma(f) = -if\}$$

$$E_1 = \{f \in \mathcal{O}_X | \sigma(f) = f\}$$

Note that $E_1 = \mathcal{O}_Y$ since it is the fixed space of G.
Let $f \in E_i$, $g \in E_{-1}$, $h \in E_{-i}$, then

$$\sigma(f^2) = \sigma(f)\sigma(f) = i^2 f^2 = -f^2 \Rightarrow f^2 \in E_{-1}$$
$$\sigma(fg) = i(-1)fg = -ifg \Rightarrow fg \in E_{-i}$$
$$\sigma(fh) = i(-i)fh = fh \Rightarrow fh \in \mathcal{O}_Y$$
$$\sigma(g^2) = (-1)^2 g^2 = g^2 \Rightarrow g^2 \in \mathcal{O}_Y$$
$$\sigma(gh) = -1(-i)gh = igh \Rightarrow gh \in E_i$$
$$\sigma(h^2) = -i^2 h^2 = -h^2 \Rightarrow h^2 \in E_{-1}$$

Thus if either E_i or E_{-i} are non-empty, neither are the rest, and each is then an invertible sheaf. In this case, $\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus E_i \oplus E_{-1} \oplus E_{-i}$. Let x, y, z be local generators for E_i, E_{-1}, E_{-i} respectively. The above computations imply:

$$x^2 = a_2 y,\, xy = a_6 z,\, xz = b_3$$

$$y^2 = b_4,\, yz = a_{13} x,\, z^2 = a_{17} y$$

Locally, this gives the map $\phi : S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$ where $\mathcal{E} \cong E_i \oplus E_{-1} \oplus E_{-i}$. Note that the traces of x, y, and z are all zero, thus $\mathcal{E}$ is the trace zero module. Also,

$$a_1 = a_3 = a_4 = a_5 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = a_{14} = a_{15} = a_{16} = a_{18} = 0.$$

Under our change of coordinates, this is requiring

$$c_{1,2} = c_{1,5} = c_{1,6} = c_{2,3} = c_{2,4} = c_{2,5} = c_{2,6} = c_{3,4} = c_{3,5} = c_{4,5} = c_{5,6} = 0,$$

leaving $c_{1,3},\ c_{1,4},\ c_{3,6},\ c_{4,6}$ free to be chosen such that the associativity relations are satisfied.

In the context of the split case, if we take $L^{-1} = E_i$, $M^{-1} = E_{-1}$, and $N^{-1} = E_{-i}$, which have weights i, -1, and $-i$ respectively with respect to the action of G, then

$$c_{1,2} \ \in \ H^0(E_i^{-2} \otimes E_{-i}) \text{ with weight } i,$$

$$c_{1,3} \ \in \ H^0(E_i^{-2} \otimes E_{-1}) \text{ with weight } 1,$$

$$c_{1,4} \ \in \ H^0(E_i^{-1} \otimes E_{-1}^{-1} \otimes E_{-i}) \text{ with weight } 1,$$

$$c_{1,5} \ \in \ H^0(E_i^{-1}) \text{ with weight } -i,$$

$$c_{1,6} \ \in \ H^0(E_i^{-1} \otimes E_{-1} \otimes E_{-i}^{-1}) \text{ with weight } -1,$$

$$c_{2,3} \ \in \ H^0(E_i^{-1}) \text{ with weight } -i,$$

$$c_{2,4} \ \in \ H^0(E_{-1}^{-2} \otimes E_{-i}) \text{ with weight } -i,$$

$$c_{2,5} \ \in \ H^0(E_{-1}^{-1}) \text{ with weight } -1,$$

$$c_{2,6} \ \in \ H^0(E_{-i}^{-1}) \text{ with weight } i,$$

$$c_{3,4} \ \in \ H^0(E_{-1}^{-1}) \text{ with weight } -1,$$

$$c_{3,5} \ \in \ H^0(E_{-i}^{-1}) \text{ with weight } i,$$

$$c_{3,6} \ \in \ H^0(E_{-1} \otimes E_{-i}^{-2}) \text{ with weight } 1,$$

$$c_{4,5} \ \in \ H^0(E_i \otimes E_{-1}^{-2}) \text{ with weight } i,$$

$$c_{4,6} \ \in \ H^0(E_i \otimes E_{-1}^{-1} \otimes E_{-i}^{-1}) \text{ with weight } 1,$$

$$c_{5,6} \ \in \ H^0(E_i \otimes E_{-i}^{-2}) \text{ with weight } -i.$$

For these to be invariant under the action of G, we must have

$$c_{1,2} = c_{1,5} = c_{1,6} = c_{2,3} = c_{2,4} = c_{2,5} = c_{2,6} = c_{3,4} = c_{3,5} = c_{4,5} = c_{5,6} = 0$$

as above.

The equations for associativity imply

$$b_3 = -c_{1,3}c_{4,6}$$

$$b_4 = c_{1,4}c_{4,6}$$

$$c_{1,3}c_{4,6} = c_{1,4}c_{3,6},$$

and this last relation is a Plücker relation in $H^0(E_i^{-1} \otimes E_{-i}^{-1})$.

Case 2 $G \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$

Let $G =< \sigma, \tau | \sigma^2 = \tau^2 = identity >$, and let

$$\begin{aligned} E_\sigma &= \{f \in \mathcal{O}_X | \sigma(f) = f, \tau(f) = -f\}, \\ E_\tau &= \{f \in \mathcal{O}_X | \sigma(f) = -f, \tau(f) = f\}, \\ E_{\sigma\tau} &= \{f \in \mathcal{O}_X | \sigma(f) = -f, \tau(f) = -f\}, \\ \mathcal{O}_Y &= \{f \in \mathcal{O}_X | \sigma(f) = f, \tau(f) = f\}. \end{aligned}$$

Let $f \in E_\sigma$, $g \in E_\tau$, $h \in E_{\sigma\tau}$, then

$$\begin{gathered} \sigma(f^2) = f^2, \tau(f^2) = f^2 \Rightarrow f^2 \in \mathcal{O}_Y, \\ \sigma(fg) = -fg, \tau(fg) = -fg \Rightarrow fg \in E_{\sigma\tau}, \\ \sigma(fh) = -fh, \tau(fh) = fh \Rightarrow fh \in E_\tau, \\ \sigma(g^2) = g^2, \tau(g^2) = g^2 \Rightarrow g^2 \in \mathcal{O}_Y, \\ \sigma(gh) = gh, \tau(gh) = -gh \Rightarrow gh \in E_\sigma, \\ \sigma(h^2) = h^2, \tau(h^2) = h^2 \Rightarrow h^2 \in \mathcal{O}_Y. \end{gathered}$$

Let x, y, and z be local coordinates for E_σ, E_τ, and $E_{\sigma\tau}$ respectively. The above computations imply:

$$x^2 = b_1, xy = a_6 z, xz = a_8 y$$

$$y^2 = b_4, yz = a_{13} x, z^2 = b_6.$$

Note again that x, y, and z have trace zero, and are therefore local generators of the trace zero module $\mathcal{E}$. Thus $\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus E_\sigma \oplus E_\tau \oplus E_{\sigma\tau}$. Now,

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_7 = a_9 = a_{10} = a_{11} = a_{12} = a_{14} = a_{15} = a_{16} = a_{17} = a_{18} = 0,$$

and thus

$$c_{1,2} = c_{1,3} = c_{1,5} = c_{2,3} = c_{2,4} = c_{2,5} = c_{2,6} = c_{3,4} = c_{3,5} = c_{3,6} = c_{4,5} = c_{5,6} = 0,$$

with $c_{1,4}$, $c_{1,6}$, $c_{4,6}$ free to be chosen so as to satisfy the relations for associativity.

In the context of the split case, if we take $L^{-1} = E_\sigma$, $M^{-1} = E_\tau$, and $N^{-1} = E_{\sigma\tau}$, which have weights σ, τ, and $\sigma\tau$ respectively under the action of G, then

$$c_{1,2} \in H^0(E_{\sigma^{-2}} \otimes E_{\sigma\tau}) \cong H^0(E_{\sigma\tau}) \text{ with weight } \sigma\tau,$$

$$c_{1,3} \in H^0(E_{\sigma^{-2}} \otimes E_{\tau}) \cong H^0(E_{\tau}) \text{ with weight } \tau,$$

$$c_{1,4} \in H^0(E_{\sigma^{-1}} \otimes E_{\tau^{-1}} \otimes E_{\sigma\tau}) \cong H^0(\mathcal{O}_Y) \text{ with weight } 1,$$

$$c_{1,5} \in H^0(E_{\sigma^{-1}}) \cong H^0(E_{\sigma}) \text{ with weight } \sigma,$$

$$c_{1,6} \in H^0(E_{\sigma^{-1}} \otimes E_{\tau} \otimes E_{(\sigma\tau)^{-1}}) \cong H^0(\mathcal{O}_Y) \text{ with weight } 1,$$

$$c_{2,3} \in H^0(E_{\sigma^{-1}}) \cong H^0(E_{\sigma}) \text{ with weight } \sigma,$$

$$c_{2,4} \in H^0(E_{\tau^{-2}} \otimes E_{\sigma\tau}) \cong H^0(E_{\sigma\tau}) \text{ with weight } \sigma\tau,$$

$$c_{2,5} \in H^0(E_{\tau^{-1}}) \cong H^0(E_{\tau}) \text{ with weight } \tau,$$

$$c_{2,6} \in H^0(E_{(\sigma\tau)^{-1}}) \cong H^0(E_{\sigma\tau} \text{ with weight } \sigma\tau,$$

$$c_{3,4} \in H^0(E_{\tau^{-1}}) \cong H^0(E_{\tau}) \text{ with weight } \tau,$$

$$c_{3,5} \in H^0(E_{(\sigma\tau)^{-1}}) \cong H^0(E_{\sigma\tau}) \text{ with weight } \sigma\tau,$$

$$c_{3,6} \in H^0(E_{\tau} \otimes E_{(\sigma\tau)^{-2}}) \cong H^0(E_{\tau}) \text{ with weight } \tau,$$

$$c_{4,5} \in H^0(E_{\sigma} \otimes E_{\tau^{-2}}) \cong H^0(E_{\sigma}) \text{ with weight } \sigma,$$

$$c_{4,6} \in H^0(E_{\sigma} \otimes E_{\tau^{-1}} \otimes E_{(\sigma\tau)^{-1}}) \cong H^0(\mathcal{O}_Y) \text{ with weight } 1,$$

$$c_{5,6} \in H^0(E_{\sigma} \otimes E_{(\sigma\tau)^{-2}}) \cong H^0(E_{\sigma}) \text{ with weight } \sigma.$$

For these to be invariant under the action of G, we must have

$$c_{1,2} = c_{1,3} = c_{1,5} = c_{2,3} = c_{2,4} = c_{2,5} = c_{2,6} = c_{3,4} = c_{3,5} = c_{3,6} = c_{4,5} = c_{5,6} = 0,$$

as is the case above.

The equations for associativity imply

$$b_1 = -c_{1,4}c_{1,6}$$

$$b_4 = c_{1,4}c_{4,6}$$

$$b_6 = -c_{1,6}c_{4,6}.$$

We summarize our work in the following theorem.

Theorem 7.2. *Let $\pi : X \to Y$ be a Galois quadruple cover.*
If π is induced by an action of $\mathbb{Z}/4\mathbb{Z}$, then

1. *$\pi_*\mathcal{O}_X$ splits into eigenspaces $\mathcal{O}_Y \oplus E_i \oplus E_{-1} \oplus E_{-i}$.*

2. *$\mathcal{E} \cong E_i \oplus E_{-1} \oplus E_{-i}$ is the trace zero module.*

3. *The totally decomposable section $\psi \in \wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$ has the local form $(c_{1,3}((x^2)^* \wedge \frac{1}{2}x^*z^*) + c_{1,4}((x^2)^* \wedge (y^2)^*) + c_{3,6}(x^*z^* \wedge (z^2)^*) + c_{4,6}((y^2)^* \wedge (z^2)^*) \otimes (x \wedge y \wedge z)$ where x, y, and z are local generators for E_i, E_{-1}, and E_{-i}, respectively, and $c_{1,3}c_{4,6} = c_{1,4}c_{3,6}$.*

4. *The multiplication map $\phi : S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$ has the form*

$$\phi(x^2) = -c_{1,3}y,$$

$$\phi(xy) = c_{1,4}z,$$

$$\phi(xz) = -c_{1,3}c_{4,6},$$
$$\phi(y^2) = c_{1,4}c_{4,6},$$
$$\phi(yz) = c_{4,6}x,$$
$$\phi(z^2) = -c_{3,6}y.$$

If π is induced by an action of $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, then

1. *$\pi_*\mathcal{O}_X$ splits into eigenspaces $\mathcal{O}_Y \oplus E_\sigma \oplus E_\tau \oplus E_{\sigma\tau}$, where σ and τ are generators for $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$.*
2. *$\mathcal{E} \cong E_\sigma \oplus E_\tau \oplus E_{\sigma\tau}$ is the trace zero module.*
3. *The totally decomposable section $\psi \in \wedge^2 S^2\mathcal{E}^* \otimes \wedge^3\mathcal{E}$ has the local form $(c_{1,4}((x^2)^* \wedge (y^2)^*) + c_{1,6}((x^2)^* \wedge (z^2)^*) + c_{4,6}((y^2)^* \wedge (z^2)^*)) \otimes (x \wedge y \wedge z)$, where $x, y,$ and z are local generators for $E_\sigma, E_\tau,$ and $E_{\sigma\tau}$, respectively.*
4. *The multiplication map $\phi : S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$ has the form*

$$\phi(x^2) = -c_{1,4}c_{1,6},$$
$$\phi(xy) = c_{1,4}z,$$
$$\phi(xz) = -c_{1,6}y,$$
$$\phi(y^2) = c_{1,4}c_{4,6},$$
$$\phi(yz) = c_{4,6}x,$$
$$\phi(z^2) = -c_{1,6}c_{4,6}.$$

7.3 Bidouble Covers

In addition, covers induced by an action of $(\mathbb{Z}/2\mathbb{Z})^2$, as in this case, have already been studied as bidouble covers. F. Catanese in [Ca] has developed this theory. We reproduce some of his work here to show the relationship between his analysis and ours.

Let $G = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and $\{\sigma, \tau, \sigma\tau\} = G - \{0\}$, $\pi : X \to Y$ the induced quadruple cover of $Y = X/G$. We may factor π as

$$\pi_\sigma : X \to X_\sigma$$
$$\pi_\tau : X \to X_\tau$$
$$\pi_{\sigma\tau} : X \to X_{\sigma\tau}$$

where $X_\sigma := X/\sigma, X_\tau := X/\tau, X_{\sigma\tau} := X/\sigma\tau$ and

$$p_\sigma : X_\sigma \to Y$$
$$p_\tau : X_\tau \to Y$$
$$p_{\sigma\tau} : X_{\sigma\tau} \to Y.$$

We have the following diagram.

$$\begin{tikzcd}
& X \arrow[dl, "\pi_\sigma"'] \arrow[d, "\pi_\tau"] \arrow[dr, "\pi_{\sigma\tau}"] & \\
X_\sigma \arrow[dr, "p_\sigma"'] & X_\tau \arrow[d, "p_\tau"] & X_{\sigma\tau} \arrow[dl, "p_{\sigma\tau}"] \\
& Y &
\end{tikzcd}$$

Let $\chi_\sigma, \chi_\tau, \chi_{\sigma\tau}$ be the non-zero characters orthogonal to σ, τ, and $\sigma\tau$, respectively. Then $(p_\sigma)_*\mathcal{O}_{X_\sigma} \cong \mathcal{O}_Y \oplus \mathcal{L}_{\chi_\sigma}$, $(p_\tau)_*\mathcal{O}_{X_\tau} \cong \mathcal{O}_Y \oplus \mathcal{L}_{\chi_\tau}$, and $(p_{\sigma\tau})_*\mathcal{O}_{X_{\sigma\tau}} \cong \mathcal{O}_Y \oplus \mathcal{L}_{\chi_{\sigma\tau}}$. The double covers $X_\sigma, X_\tau, X_{\sigma\tau}$ are ramified on $D_\tau + D_{\sigma\tau}, D_\sigma + D_{\sigma\tau}, D_\sigma + D_\tau$, respectively. Thus we may choose local equations $x_\sigma, x_\tau, x_{\sigma\tau}$ of $D_\sigma, D_\tau, D_{\sigma\tau}$ such that $x_\tau x_{\sigma\tau}$ gives the homomorphism

$$\mathcal{L}_{\chi_\sigma}^{\otimes 2} \to \mathcal{O}_Y,$$

and thus the algebra structure to $(p_\sigma)_*\mathcal{O}_{X_\sigma}$ over $\mathcal{O}_Y$, $x_\sigma x_{\sigma\tau}$ gives the homomorphism

$$\mathcal{L}_{\chi_\tau}^{\otimes 2} \to \mathcal{O}_Y,$$

and thus the algebra structure to $(p_\tau)_*\mathcal{O}_{X_\tau}$ over $\mathcal{O}_Y$, and $x_\sigma x_\tau$ gives the homomorphism

$$\mathcal{L}_{\chi_{\sigma\tau}}^{\otimes 2} \to \mathcal{O}_Y,$$

and thus the algebra structure to $(p_{\sigma\tau})_*\mathcal{O}_{X_{\sigma\tau}}$ over $\mathcal{O}_Y$.

Thus we have $\mathcal{L}_{\chi_\sigma} \cong \mathcal{O}_Y(-L_\sigma)$, $\mathcal{L}_{\chi_\tau} \cong \mathcal{O}_Y(-L_\tau)$, and $\mathcal{L}_{\chi_{\sigma\tau}} \cong \mathcal{O}_Y(-L_{\sigma\tau})$, with

$$2L_\sigma \equiv D_\tau + D_{\sigma\tau} \tag{7.3}$$
$$2L_\tau \equiv D_\sigma + D_{\sigma\tau} \tag{7.4}$$
$$2L_{\sigma\tau} \equiv D_\sigma + D_\tau \tag{7.5}$$

.

From equations 7.3, 7.4, and 7.5, we see that

$$2(L_\sigma + L_\tau - L_{\sigma\tau}) \equiv 2D_{\sigma\tau},$$
$$2(L_\tau + L_{\sigma\tau} - L_\sigma) \equiv 2D_\sigma,$$
$$2(L_\sigma + L_{\sigma\tau} - L_\tau) \equiv 2D_\tau.$$

Therefore $x_{\sigma\tau}$ is a section of $\mathcal{O}_Y(-L_\sigma) \otimes \mathcal{O}_Y(-L_\tau) \otimes \mathcal{O}_Y(L_{\sigma\tau})$, i.e. a homomorphism

$$\mathcal{O}_Y(-L_\sigma) \otimes \mathcal{O}_Y(-L_\tau) \to \mathcal{O}_Y(-L_{\sigma\tau}).$$

Similarly, x_σ is a section of $\mathcal{O}_Y(-L_\tau) \otimes \mathcal{O}_Y(-L_{\sigma\tau}) \otimes \mathcal{O}_Y(L_\sigma)$, i.e. a homomorphism

$$\mathcal{O}_Y(-L_\tau) \otimes \mathcal{O}_Y(-L_{\sigma\tau}) \to \mathcal{O}_Y(-L_\sigma),$$

and x_τ is a section of $\mathcal{O}_Y(-L_\sigma) \otimes \mathcal{O}_Y(-L_{\sigma\tau}) \otimes \mathcal{O}_Y(L_\tau)$, i.e. a homomorphism

$$\mathcal{O}_Y(-L_\sigma) \otimes \mathcal{O}_Y(-L_{\sigma\tau}) \to \mathcal{O}_Y(-L_\tau).$$

Now we see that the choices of sections $x_\sigma, x_\tau, x_{\sigma\tau}$ are the choices we have seen above in the non-zero Plücker coordinates

$$c_{1,4} \in H^0(E_{\sigma^{-1}} \otimes E_{\tau^{-1}} \otimes E_{\sigma\tau}),$$
$$c_{4,6} \in H^0(E_\sigma \otimes E_{\tau^{-1}} \otimes E_{(\sigma\tau)^{-1}}),$$

and

$$c_{1,6} \in H^0(E_{\sigma^{-1}} \otimes E_\tau \otimes E_{(\sigma\tau)^{-1}}).$$

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