COMPONENT NUMBERS FOR TORSION SECTIONS OF SEMISTABLE ELLIPTIC SURFACES

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Introduction

In this article we would like to address the following situation. Let $f : X \to C$ be an elliptic surface with section S_0. We further assume that X is relatively minimal and smooth, and that all singular fibers are semistable, that is, are cycles of $\mathbb{P}^1$'s (type "I_m" in Kodaira's notation, see [K]). Given a section S of f, it will meet one and only one component in every singular fiber. The fundamental aim of this paper is to try to say something specific about which components are being hit, in the case that S is a torsion section (i.e., of finite order in the Mordell-Weil group of sections of f).

Most work on the Mordell-Weil group for elliptic surfaces focuses on properties of the group itself, (for example its rank, etc.) and not on properties of the elements of the group. See for example [Shd1] and [C-Z] for general properties, and papers such as [Sch] and [Stl] for computations in specific cases. Although the methods used in this article are elementary, they are suprisingly powerful, enabling us to obtain rather detailed information about the possibilities for component intersection for a torsion section.

To be specific, suppose that the j^{th} singular fiber is a cycle of length m_j, and that the components are numbered cyclically around the cycle from 0 to $m_j - 1$, with the component numbered 0 being the one meeting the zero section S_0. (Note that there are two ways to do this, depending on the two orderings around the cycle.) Suppose that in this j^{th} fiber, a section S of order exactly n hits the k_j^{th} component. The main restrictions on these "component" numbers k_j are the following:

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The Quadratic Relation (Proposition 3.1):

$$\sum_j k_j(1 - k_j/m_j) = 2\chi(\mathcal{O}_X).$$

The Component Number Sum Relation (Corollaries 4.1 and 4.4): If the component numbers for S satisfy $k_j \leq m_j/2$, (which can always be arranged by the suitable "reorienting" of the cycle), then

$$\sum_{j=1}^{s} k_j(S) = \begin{cases} 4\chi(\mathcal{O}_X) & \text{if } n = 2 \\ 3\chi(\mathcal{O}_X) & \text{if } n \geq 3. \end{cases}$$

The Quadratic Relation as stated above follows directly (as a special case) from Shioda's construction of a height pairing on the Mordell-Weil group, see [Shd2, Theorem 8.6 and Table 8.16] and [Shd3, Proposition 3.5(iv)]. We give a direct proof which is not different from Shioda's in any essential way, but which is arranged to arrive at this one formula.

These relations hold for the general torsion section, but actually we have more to say in the case of a torsion section S of prime order p. In this case we are able to calculate exactly the distribution of the "distances" between the components hit by S and the components hit by the zero-section S_0, in the following sense. If a fiber has m components, indexed from 0 to $m-1$ cyclically, then S, having order p, meets either the component indexed by 0 (which we take to be the component meeting S_0) or one of the components indexed by im/p for some i. In other words, a section of order p must meet either $1/p$ of the way around the cycle, or $2/p$ of the way around, etc. Denote by $M_{i,p}(S)$ the fraction of the total Euler number $e = \sum_j m_j$ contributed by those fibers in which S meets i/p of the way around the cycle, in either direction. Our main result is a computation of these numbers, which turn out to be independent of S, and surprisingly, independent of i (as long as $i \neq 0$):

The Distribution Numbers (Theorem 6.3):
Let S be a torsion section of odd prime order p. Then

$$M_{i,p}(S) = 2p/(p^2 - 1) \ \text{ if } \ 1 \leq i \leq (p-1)/2,$$

and

$$M_{0,p}(S) = 1/(p+1).$$

Thus we obtain an "equidistribution" of these distances: the total sum of those m_j's contributed from fibers of type I_{m_j} in which S meets i/p of the way around the cycle is constant, independent of i.

Thus a torsion section of prime order, as it travels across the elliptic surface, is forced to be completely "fair" in its choice of which components it may hit, in the above sense.

The above results generalize various statements concerning the component numbers $\{k_j\}$ for a torsion section which appeared in [M-P1], [M-P2], [M-P3], and [M].

The results in this paper could be proved by appealing to the appropriate elliptic modular surface (for those primes where such exists) and its universal properties. The equidistribution property is invariant under base change, so if it is true for the (universal) modular surface, it will be true in general. This is a computation involving the modular group, which could be made once and for all. However it is my intention to show that these properties are somehow more elementary than the theory of modular surfaces, and follow from straightforward considerations involving only basic facts concerning the Mordell-Weil group and a little intersection theory of surfaces.

In a further paper we hope to discuss the case of torsion of non-prime order more fully.

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1. Basic Facts

We work over the field of complex numbers. Let C be a smooth curve, and let $f : X \to C$ be a smooth semistable elliptic surface over C. We suppose that f has a chosen section S_0 which we treat as the zero for the group law on the sections of f. Let g be the genus of C, and let $\chi = \chi(\mathcal{O}_X)$ be the holomorphic Euler characteristic of X. If e is the topological Euler number of X, then $e = 12\chi$.

We suppose that the map f has s singular fibers $F_1, \ldots, F_s$, each semistable, i.e., each a cycle of rational curves (of type "I_m" in Kodaira's notation [K]). Indeed, let us say that the fiber F_j is of type I_{m_j}. Therefore

$$e = 12\chi = \sum_{j=1}^{s} m_j.$$

Choose an "orientation" of each fiber F_j and write the m_j components of F_j as

$$C_0^{(j)}, C_1^{(j)}, \dots, C_{m_j-1}^{(j)},$$

where the zero section S_0 meets only $C_0^{(j)}$ and for each k, $C_k^{(j)}$ meets only $C_{k\pm1}^{(j)} \mod m_j$. If $m_j = 1$, then $F_j = C_0^{(j)}$ is a nodal rational curve, of self-intersection 0. If $m_j \geq 2$, then each $C_k^{(j)}$ is a smooth rational curve with self-intersection -2.

If we denote by $\mathrm{NS}(X)$ the Neron-Severi group of the elliptic surface X, we can consider the sublattice $L \subseteq \mathrm{NS}(X)$ generated by the zero-section S_0 and the components $C_k^{(j)}$ of the singular fibers. The class of the fiber F is of course in this sublattice L; moreover, the only relations among these classes are that

$$F \equiv \sum_{k=0}^{m_j-1} C_k^{(j)} \text{ for each } j = 1, \dots, s. \tag{1.1}$$

If we let U denote the sublattice of L spanned by F and S_0, then U is a unimodular lattice of rank 2, isomorphic to a hyperbolic plane, and so splits off L. Its orthogonal complement R is freely generated by the components $C_k^{(j)}$ for $j = 1 \dots s$ and $k \neq 0$. R is therefore a direct sum of s root lattices of type A, with the j^{th} summand isomorphic to A_{m_j-1}, generated by $C_k^{(j)}$ for $k \neq 0$.

The Shioda-Tate formula for the Mordell-Weil group $\mathrm{MW}(X)$ of sections of X is derived from the exact sequence (see [Shd1])

$$0 \to L \to \mathrm{NS}(X) \to \mathrm{MW}(X) \to 0$$

where the first map is the inclusion of the sublattice L into $\mathrm{NS}(X)$. The second map is the fiber-by-fiber summation map, sending a divisor class $D \in \mathrm{NS}(X)$ to the closure of the sum of the points of D on the generic fiber of X. We therefore obtain information about both the rank and the torsion of the Mordell-Weil group $\mathrm{MW}(X)$. If we denote by ρ the Picard number of X, which is the rank of the Neron-Severi group, we see that since L has rank equal to $2 + \sum_{j=1}^{s}(m_j - 1)$,

$$\operatorname{rank} \mathrm{MW}(X) = \rho - 2 - \sum_{j=1}^{s}(m_j - 1) = s + \rho - 2 - e.$$

The torsion $\mathrm{MW}_{tor}(X)$ in the Mordell-Weil group corresponds to those classes in $\mathrm{NS}(X)$ for which some multiple lies in the sublattice L. These classes form an intermediate lattice L^{sat} containing L as a sublattice of finite index, and we see that

$$\mathrm{MW}_{tor}(X) \cong L^{sat}/L.$$

For any lattice N, denote by $N^{\#}$ the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$. The inclusion of L into L^{sat} gives the sequence of inclusions

$$L \subseteq L^{sat} \subseteq (L^{sat})^{\#} \subseteq L^{\#}$$

which shows that the quotient group L^{sat}/L is isomorphic to a subgroup of the discriminant-form group $G_L = L^{\#}/L$. Note that $L^{\#} = U^{\#} \oplus R^{\#}$ since $L = U \oplus R$ and since $U^{\#} = U$ (U is unimodular), we have $G_L \cong R^{\#}/R = \bigoplus_{j=1}^{s} A^{\#}_{m_j-1}/A_{m_j-1}$. A computation shows that for the root lattices of type A, $A^{\#}_m/A_m \cong \mathbb{Z}/m\mathbb{Z}$; therefore we have that the torsion part $\mathrm{MW}_{tor}(X)$ of the Mordell-Weil group of sections of X is isomorphic to a subgroup of $\bigoplus_{j=1}^{s} \mathbb{Z}/m_j\mathbb{Z}$. (See for example [M-P1] or [M].)

A choice of orientation for each singular fiber F_j gives an identification of the cyclic group of components $C^{(j)}_k$ for $k = 0, \ldots, m_j - 1$ with $\mathbb{Z}/m_j\mathbb{Z}$. For any section S of f, and any singular fiber F_j, denote by $k_j(S)$ the index of the component of F_j which S meets. Thus

$$S \cdot C^{(j)}_l = \delta_{l k_j(S)} \tag{1.2}$$

where δ above is the Kronecker delta. Note that with our notation above, $k_j(S_0) = 0$ for every j. The numbers $\{k_j(S)\}$ will be called the *component numbers* of the section S. The upshot of the remarks above is that, for torsion sections, this assignment of integers $\{k_j(S)\}$ to S is 1-1. Thus a torsion section is determined by its component numbers (but not all sets of components numbers can occur).

It is the purpose of this article to study the properties of the component numbers of torsion sections for semistable elliptic surfaces. Although it is clear from the above discussion that the component numbers $k_j(S)$ are well-defined modulo m_j, it is more useful for our purposes to take them to be integers in the range $0, \ldots, m_j - 1$.

Two facts are necessary for what follows; they can be found in [M] as well as the other basic references.

Lemma 1.3.

(1.3.1) If S_1 and S_2 are two different sections in $\mathrm{MW}(X)$ *with $S_1 - S_2$ torsion, then $S_1 \cdot S_2 = 0$, i.e., they are disjoint.*

(1.3.2) If S is any section in $\mathrm{MW}(X)$*, then $S \cdot S = -\chi$.*

2. The Divisor Class of a Torsion Section

Let S be a torsion section of the semistable elliptic surface $f : X \to C$. In this section we wish to write down the class of S in $\mathrm{NS}(X)$.

Since the class of S lies in L^{sat}, we have that S is a $\mathbb{Q}$-linear combination of the zero section S_0, and the fiber components $C^{(j)}_k$. Because

(1.1) is the only relation among these classes, the classes S_0, F, and $C_k^{(j)}$ for $j = 1, \dots, s$ and $k \neq 0$ form a basis for the module spanned by all these classes.

For fixed $j = 1, \dots, s$ and $k = 1, \dots, m_j - 1$, set

$$\begin{aligned} D_k^{(j)} &= (m_j - k)\sum_{i=1}^{k} iC_i^{(j)} + k\sum_{i=k+1}^{m_j-1}(m_j - i)C_i^{(j)} \\ &= (m_j - k)[C_1^{(j)} + 2C_2^{(j)} + \cdots + kC_k^{(j)}] + \\ &\quad k[(m_j - k - 1)C_{k+1}^{(j)} + \cdots + 2C_{m_j-2}^{(j)} + C_{m_j-1}^{(j)}]. \end{aligned}$$

We set $D_0^{(j)} = 0$.

Note that

$$D_k^{(j)} \cdot S_0 = 0 \tag{2.1}$$

for every k.

Lemma 2.2. *Fix $j = 1, \dots, s$, and indices k and l with $0 \leq k, l \leq m_j - 1$. Then*

$$D_k^{(j)} \cdot C_l^{(j)} = \begin{cases} m_j & \text{if } l = 0 \text{ and } k \neq 0 \\ -m_j & \text{if } l = k \text{ and } k \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is a straightforward check; let us first notice that if $k = 0$, then $D_k^{(j)} = 0$, so certainly the intersection product is 0. Assume then that $k \neq 0$. If $l = 0$, then since $C_0^{(j)}$ meets only $C_1^{(j)}$ and $C_{m_j-1}^{(j)}$, each once, we have

$$D_k^{(j)} \cdot C_0^{(j)} = (m_j - k)[1] + (k)[1] = m_j,$$

which proves the first statement. If $l = k$, then $C_k^{(j)}$ meets only $C_{k\pm1}^{(j)}$ (each once) and itself (-2 times), so that

$$D_k^{(j)} \cdot C_k^{(j)} = (m_j - k)[(k-1)(1) + (k)(-2)] + (k)[(m_j - k - 1)(1)] = -m_j,$$

proving this case. We leave to the reader to check that all other intersection products are 0, as stated. $\square$

It is useful to note that the above lemma can be re-expressed using the Kronecker delta function as

$$D_k^{(j_1)} \cdot C_l^{(j_2)} = m_j \delta_{j_1 j_2}(1 - \delta_{k0})(\delta_{l0} - \delta_{lk}). \tag{2.3}$$

Theorem 2.4. *With the above notation, if S is a torsion section, then*

$$S \equiv S_0 + (S - S_0 \cdot S_0)F - \sum_{j=1}^{s} \frac{1}{m_j} D_{k_j(S)}^{(j)}.$$

Proof. Since the intersection form on L is nondegenerate, we need only check that both sides of the equation intersect the generators S_0, F, and the $C_l^{(j)}$ in the same number. Let us begin with S_0. Using (2.1) and the equations

$$S \cdot F = S_0 \cdot F = 1,$$

we have that the right hand side intersects S_0 to $S \cdot S_0$, agreeing with the left hand side. Similarly, F meets only the S_0 term on the right hand side, so the intersections with F are also equal.

Now fix indices $j = 1, \ldots, s$ and $l = 0, \ldots, m_j - 1$, and let us check the intersection with $C_l^{(j)}$; Lemma 2.2 is the critical part of the computation. Intersecting with the right hand side gives $S_0 \cdot C_l^{(j)} - \frac{1}{m_j} D_{k_j(S)}^{(j)} \cdot C_l^{(j)}$, which reduces to $\delta_{l0} - (1 - \delta_{k_j(S)0})(\delta_{l0} - \delta_{lk_j(S)})$ using (2.3). This is after some simplification equal to $\delta_{lk_j(S)}$, which is also $S \cdot C_l^{(j)}$ by (1.2). $\square$

The reader familiar with [C-Z] will recognize the above expression as the "correction" term in their pairing on the Mordell-Weil group. They were more interested in questions of rank in that paper, and less so in the torsion sections.

The formula for the linear equivalence class of S given above can be simplified somewhat in the case that S is not the zero section S_0. Then $(S \cdot S_0) = 0$ and $(S_0 \cdot S_0) = -\chi$ by Lemma 1.3, so we have

$$S \equiv S_0 + \chi F - \sum_{j=1}^{s} \frac{1}{m_j} D_{k_j(S)}^{(j)}. \tag{2.5}$$

3. The Quadratic Relation for the Component Numbers

Let us now take the above formula for the torsion section S and intersect it with S itself. If $S = S_0$, then of course all $k_j(S) = 0$, both sides of the equation are S_0, so we recover no information. However if $S \neq S_0$, the formula (2.5) is nontrivial. Dotting S with the right-hand side gives

$$\chi - \sum_{j=1}^{s} \frac{1}{m_j} D_{k_j(S)}^{(j)} \cdot S.$$

Since S meets only the curve $C_{k_j(S)}^{(j)}$ in the j^{th} singular fiber, and it meets it exactly once, we have

$$D_{k_j(S)}^{(j)} \cdot S = (m_j - k_j(S)) k_j(S).$$

Therefore the above reduces to

$$\chi - \sum_{j=1}^{s} k_j(S)(1 - \frac{k_j(S)}{m_j}).$$

Finally, using Lemma 1.3.2, dotting the left-hand side with S gives $-\chi$. Hence we obtain the following formula, which we call the *quadratic relation* for the component numbers.

Proposition 3.1. *Let S be a torsion section of f, not equal to the zero section S_0. Then*

$$\sum_{j=1}^{s} k_j(S)(1 - \frac{k_j(S)}{m_j}) = 2\chi(\mathcal{O}_X).$$

Note that the quadratic relation is independent of the choice of orientation of each singular fiber, as it should be. (If one reverses the orientation of F_j, then k_j is replaced by $m_j - k_j$ if $k_j \neq 0$, so that the two terms of each summand are simply switched.)

As we noted in the Introduction, the quadratic relation is also a consequence of Shioda's general theory of the height pairing on the Mordell-Weil group. Miles Reid has pointed out to me that the quadratic relation given above can be derived directly via the use of the Riemann-Roch theorem for Weil divisors, as expounded in [R, section 9]. One should contract all components of fibers except $C_0^{(j)}$, obtaining a surface with only A_{m_j-1} singularities, and consider on that surface the Weil divisor S. The Riemann-Roch formula for S has as corrections to the usual Riemann-Roch the terms $k_j(S)(1-\frac{k_j(S)}{m_j})$. (See also [G, section 2].) This approach is cleaner and perhaps conceptually simpler; however in the spirit of keeping things as elementary as possible, I have decided to present this "low road" version. The interested reader will have no problem making the computation as Reid suggests and re-deriving the quadratic relation in this manner.

4. The Component Number Sums

In this section we will develop formulas for the component number sums $\sum_j k_j(S)$ for a torsion section S. Such formulas follow rather directly from the quadratic relation for the component numbers. Suppose first that S has order 2. Then for each j with $k_j(S) \neq 0$, we must have m_j even and $k_j(S) = m_j/2$. Therefore the quadratic relation reduces to the following.

Corollary 4.1. *Let S be an order 2 section of f. Then*

$$\sum_{j=1}^{s} k_j(S) = 4\chi(\mathcal{O}_X).$$

Note that in the order 2 case, each k_j is either 0 or $m_j/2$, and hence the component numbers for a torsion section of order 2 are independent of the orientation of the fibers. This is clearly not true for other values of the component numbers. In general, reversing the orientation of the fiber F_j changes the component number from k_j to $m_j - k_j$. Hence the analogue of Corollary (4.1) for torsion sections of order at least 3 must take this into account.

To this end define a function $d_m : \{0, \dots, m-1\} \to \{0, \dots, [m/2]\}$ by setting

$$d_m(k) = \min\{k, m-k\}.$$

Note that for any given torsion section S, it is possible to choose the orientations of the fibers so that the component numbers $k_j(S)$ are minimal, that is, $k_j(S) = d_{m_j}(k_j(S))$ for each j. If this condition holds, we say that the section S has *minimal component numbers*.

Next suppose that a torsion section S has order $n \geq 3$. Then both S and $2S$ are nonzero torsion sections of X. If for each index j, we choose the orientation of the components so that $0 \leq k_j(S) \leq m_j/2$, then we have the formulas

$$k_j(2S) = \begin{cases} 0 \text{ if } k_j(S) = m_j/2, \text{ and} \\ 2k_j(S) \text{ if } k_j(S) < m_j/2 \end{cases} \tag{4.2}$$

for every j. Therefore applying Proposition 3.1 to $2S$, after dividing by 2 we obtain

$$\sum_{\substack{j \text{ with} \\ k_j(S) < m_j/2}} k_j(S)(1 - 2\frac{k_j(S)}{m_j}) = \chi(\mathcal{O}_X).$$

Multiplying the formula of Proposition 3.1 by 2 gives

$$\sum_{\substack{j \text{ with} \\ k_j(S) = m_j/2}} k_j(S) + \sum_{\substack{j \text{ with} \\ k_j(S) < m_j/2}} k_j(S)(2 - 2\frac{k_j(S)}{m_j}) = 4\chi(\mathcal{O}_X),$$

and subtracting the above two equations yields the following.

Corollary 4.3. *Let S be a section of order $n \geq 3$ with minimal component numbers $\{k_j(S)\}$. Then*

$$\sum_{j=1}^{s} k_j(S) = 3\chi(\mathcal{O}_X).$$

In case the orientations of the fibers are not chosen so as to give S minimal component numbers, a similar formula holds, expressed with the d_m function.

Corollary 4.4. *Let S be a section of order $n \geq 3$. Then*

$$\sum_{j=1}^{s} d_{m_j}(k_j(S)) = 3\chi(\mathcal{O}_X).$$

Using these component number sum relations, one can re-express the quadratic relation in a simpler form.

Corollary 4.5. *Let S be a section of order $n \geq 2$, with minimal component numbers $\{k_j = k_j(S)\}$. Then*

$$\sum_{j=1}^{s} k_j^2/m_j = \begin{cases} \chi(\mathcal{O}_X) & \text{if } n \geq 3 \\ 2\chi(\mathcal{O}_X) & \text{if } n = 2 \end{cases}$$

Proof. The quadratic relation can be written as

$$\sum_j k_j^2/m_j = \sum_j k_j - 2\chi(\mathcal{O}_X),$$

and so the result follows from Corollaries 4.1 and 4.4 $\square$

These results generalize a certain divisibility statement concerning the component numbers obtained in [M-P1, Proposition 4.6] (see also [M-P2, Lemma 3(a)] and [M, X.3.1]). Although the above cited works deal with elliptic K3 surfaces, the statement holds in general. It is:

$$\sum_{j=1}^{s} \frac{m_j - 1}{2m_j} k_j^2 \text{ is an integer.}$$

This was proved using lattice-theoretic arguments involving the discriminant-form group $L^{\#}/L$.

We can recover this result as follows. First note that in the above sum, if any k_j is replaced by $m_j - k_j$, the sum changes by an integer; hence it suffices for us to derive the divisibility result using minimal component numbers.

Now the above sum in question can be rewritten as

$$\sum_j k_j^2/2 - \sum_j k_j^2/2m_j.$$

Note that $\sum_j k_j^2$ has the same parity as $\sum_j k_j$, so that the first term $\sum_j k_j^2/2$ is equal to $\sum_j k_j/2$ modulo $\mathbb{Z}$. Now modulo $\mathbb{Z}$, we have

$$\begin{aligned}
\sum_j k_j^2/2 - \sum_j k_j^2/2m_j &= \sum_j k_j/2 - \sum_j k_j^2/2m_j \\
&= \begin{cases} 4\chi/2 - 2\chi/2 & \text{if } n = 2 \\ 3\chi/2 - \chi/2 & \text{if } n \geq 3 \end{cases} \\
&= \chi
\end{aligned}$$

which is an integer.

5. The Distribution Numbers for a Torsion Section

In this section we will apply the relations developed in the previous sections for the component numbers of a non-zero torsion section S to its multiples αS. To this end let us express the quadratic relation by using the following notation. For each singular fiber F_j, denote by $f_j(S)$ the fraction $k_j(S)/m_j$. Let $P(x) = x(1-x)$ for real numbers x. Then the quadratic relation can be expressed as

$$\sum_{j=1}^{s} P(f_j(S))\, m_j = 2\chi(\mathcal{O}_X). \tag{5.1}$$

Note again that this formula is independent of the choice of orientation of the fibers, since changing the orientation of the components of F_j simply switches the two factors f_j and $1 - f_j$ in each P term of the above sum.

Now note that if one fixes orientations for all of the singular fibers, and if a torsion section S has component numbers $\{k_j(S)\}$ using these orientations, then for any α, the component numbers for the multiple αS of S in these orientations will be

$$k_j(\alpha S) \equiv \alpha k_j(S) \mod m_j.$$

Let $\langle x \rangle$ denote the smallest non-negative real number in the residue class modulo $\mathbb{Z}$ of a real number x; we have $\langle x \rangle \in [0, 1)$ always. The above formula can be expressed using the fractions f_j by

$$f_j(\alpha S) = \langle \alpha f_j(S) \rangle.$$

Hence the quadratic relation (5.1) applied to the section αS is

$$\sum_{j=1}^{s} P(\langle \alpha f_j(S) \rangle) m_j = 2\chi(\mathcal{O}_X). \tag{5.2}$$

We want to reorganize this sum to collect terms with the same f_j. Let S be any torsion section whose order divides n. Then nS is the zero-section S_0, all of whose component numbers are zero; hence for each j, $nk_j(S)$ is divisible by m_j. In terms of the fractions f_j, this is equivalent to having $f_j(S)$ being a multiple of $1/n$ for every j.

Define rational numbers ${M'}_{i,n}(S)$ for $0 \leq i \leq n-1$ by the formula

$$ {M'}_{i,n}(S) = (\sum_{\substack{j \text{ with} \\ f_j(S)=i/n}} m_j)/12\chi. $$

Roughly speaking, ${M'}_{i,n}(S)$ is the fraction of the total sum $\sum_j m_j = 12\chi$ contributed by fibers where S meets the component which is "distance i" from the component meeting the zero-section S_0. (This distance is measured in units of m_j/n, and is in the direction of the orientation of the fiber.) We will call these fractions ${M'}_{i,n}(S)$ the *oriented distribution numbers* for the section S. Obviously

$$ \sum_{i=0}^{n-1} {M'}_{i,n}(S) = 1. $$

Reorganizing the quadratic relation (5.2) for αS to sum over the i's gives

$$ \sum_{i=0}^{n-1} \sum_{\substack{j \text{ with} \\ f_j(S)=i/n}} P(\langle \alpha i/n \rangle) m_j = 2\chi(\mathcal{O}_X), $$

which, after dividing through by $12\chi(\mathcal{O}_X)$, we rewrite using the oriented distribution numbers as

$$ \sum_{i=0}^{n-1} P(\langle \alpha i/n \rangle) {M'}_{i,n}(S) = 1/6. $$

If we disregard orientation, we are led to defining rational numbers $M_{i,n}(S)$ for $0 \leq i \leq [n/2]$ by

$$ M_{i,n}(S) = \begin{cases} {M'}_{0,n}(S) & \text{if } i = 0, \\ {M'}_{i,n}(S) + {M'}_{n-i,n}(S) & \text{if } 1 \leq i < n/2, \text{ and} \\ {M'}_{n/2,n}(S) & \text{if } i = n/2. \end{cases} $$

Note that $M_{i,n}(S)$ is the fraction of the total sum $\sum_j m_j = 12\chi$ contributed by fibers where S meets the component which is "distance i" from the component meeting the zero-section S_0, in either of the two orientations. We therefore refer to the $M_{i,n}(S)$ as the *unoriented distribution numbers* for the section S. As with the oriented distribution numbers, we have

$$ \sum_{i=0}^{[n/2]} M_{i,n}(S) = 1. \tag{5.3} $$

Since $\langle -x\rangle = 1 - \langle x\rangle$, and $\langle \ell + x\rangle = \langle x\rangle$ for integers ℓ, we see that $\langle \alpha(n-i)/n\rangle = 1 - \langle \alpha i/n\rangle$. Therefore since the polynomial $P(x)$ has the same value at x as at $1-x$, the quadratic relation written in terms of the oriented distribution numbers has the coefficient of $M'_{i,n}(S)$ equal to the coefficient of $M'_{n-i,n}(S)$. Hence we may combine these terms, obtaining the following relation for the unoriented distribution numbers. (Note that the $i = 0$ term has been dropped, since it is zero.)

Lemma 5.4. *If S is a torsion section of order exactly n, and α is not divisible by n, then*

$$\sum_{i=1}^{[n/2]} P(\langle \alpha i/n\rangle) M_{i,n}(S) = 1/6.$$

Note that one gets the same equation relating the unoriented distribution numbers $M_{i,n}(S)$ using either α or $n - \alpha$. Hence the above set of equations are, a priori, only $[n/2]$ different equations.

One can express the above set of equations in matrix form. Let $\mathbf{P}_n$ be the square matrix of size $[n/2]$ whose αi^{th} entry is $P(\langle \alpha i/n\rangle)$. Let $\mathbf{1}_n$ be a vector of length $[n/2]$, which has every coordinate equal to 1. Finally let $\mathbf{M}_n(S)$ be the column vector of $M_{i,n}(S)$'s, for $1 \le i \le [n/2]$. The equations of 5.4 can then be expressed in matrix form as follows.

Lemma 5.5. *If S is a nonzero torsion section of order exactly n, then*

$$\mathbf{P}_n \mathbf{M}_n(S) = (1/6)\mathbf{1}_n.$$

For example, suppose that $n = 2$, so that we are discussing a 2-torsion section S of X. The linear system above reduces to the single equation $(1/2)M_{1,2}(S) = 1/6$, so that $M_{1,2}(S) = 1/3$ and therefore $M_{0,2}(S) = 2/3$ by (5.3).

In the case of $n = 3$, the linear system again reduces to the single equation $(2/9)M_{1,3}(S) = 1/6$, so that $M_{1,3}(S) = 3/4$ and therefore $M_{0,3}(S) = 1/4$ by (5.3).

Note that these distribution numbers are independent of S! This is our result in general for p-torsion sections, which will be discussed in the next section.

As a last example, suppose that $n = 5$. Then the linear system above is the two equations $(4/25)M_{1,5}(S) + (6/25)M_{2,5} = 1/6$ and $(6/25)M_{1,5}(S) + (4/25)M_{2,5} = 1/6$ which leads to $M_{1,5}(S) = M_{2,5}(S) = 5/12$, and $M_{0,5}(S) = 1/6$. Note again that the distribution numbers are independent of S, as we have seen in the previous examples. However here we see more: the distribution numbers $M_{i,5}$ for $i \neq 0$ are *equal*. This is the "equidistribution" property mentioned in the Introduction, and is our main result for general primes p.

6. Equidistribution for Torsion Sections of Prime Order

In this section we will use the linear system describing the unoriented distribution numbers $M_{i,n}(S)$ to compute these numbers for a torsion section S of prime order p.

Our first task is to show that the matrix $\mathbf{P}_p$ is invertible, which therefore shows that the distribution numbers are determined by the linear system of Lemma 5.5.

Proposition 6.1. *Fix an odd prime p. Let $\mathbf{P}_p$ be the square matrix of size $(p-1)/2$ whose $\alpha\, i$ entry is $P(\langle \alpha i/p\rangle)$. Then $\mathbf{P}_p$ is invertible.*

Proof. Let V be $\mathbb{C}^p$, with coordinates indexed by residue classes modulo p. A vector $v \in V$ will have as its k^{th} coordinate the number v_k. The group of units $G = (\mathbb{Z}/p)^\times$ acts on V, by the formula

$$(a \cdot v)_k = v_{ak \mod p}$$

for $a \in G$.

Define vectors $S_\alpha \in V$ by setting $(S_\alpha)_k = P(\langle \alpha k/p\rangle)$. We obviously have $S_\alpha = S_{p-\alpha}$. Also note that $(S_\alpha)_0 = 0$ for every α, and $(S_\alpha)_{p-k} = (S_\alpha)_k$ for every α and k.

Therefore S_α lies in the subspace V^+ defined by

$$V^+ = \{v \in V \mid v_0 = 0 \text{ and } v_{p-k} = v_k \text{ for all } k\}.$$

The dimension of V^+ is $(p-1)/2$. Moreover V^+ is stable under the G-action, so that we can consider G to be acting on V^+ also, with the same formula. Further note that $a \cdot S_1 = S_a$ for every $a \in G$.

Note that the α^{th} row of $\mathbf{P}_p$ consists of the coordinates $(S_\alpha)_k$ for $k = 1, \dots, (p-1)/2$; hence to prove that $\mathbf{P}_p$ is invertible, it suffices to show that the vectors S_α, for $\alpha = 1, \dots, (p-1)/2$, are independent. Equivalently, we can show that the vectors $\{S_\alpha\}$ span V^+.

Let $\chi : G \to \mathbb{C}^\times$, be a character of G. For $v \in V^+$, define

$$\omega_\chi(v) = \sum_{a \in G} \chi(a)\, a \cdot v.$$

Note that $\omega_\chi(v)$ is an eigenvector of the action of G on V^+, with eigenvalue χ^{-1}, that is, $a \cdot \omega_\chi(v) = \chi^{-1}(a)\omega_\chi(v)$.

Let us focus in on the special element

$$S_\chi = \omega_\chi(S_1) = \sum_{a \in G} \chi(a) S_a.$$

Since this vector is a linear combination of the S_a's, to show that the S_a's span V^+, it suffices to show that the S_χ's do.

We focus attention on the *even* characters χ, that is, those characters with $\chi(-1) = 1$. (If $\chi(-1) = -1$, χ is called *odd*.) There are exactly

$(p-1)/2$ even characters for G, and we will finish the proof if we can show that for the $(p-1)/2$ even characters χ, the vectors S_χ are non-zero; if so, they will be linearly independent, since they are eigenvectors for distinct characters. Hence they will span V^+.

In fact we need only show that the first coordinate of each S_χ (for χ even) is non-zero. This is the number

$$s_\chi = (S_\chi)_1 = \sum_{a \in G} \chi(a) P(\langle a/p \rangle).$$

If χ is the trivial character which is identically 1, then s_χ is obviously non-zero, so we may assume χ is one of the $(p-3)/2$ nontrivial even characters of G.

Let $\mathbf{B}_k(X)$ denote the k^{th} Bernoulli polynomial. In particular, we have

$$\mathbf{B}_2(X) = X^2 - X + 1/6 = 1/6 - P(X), \text{ or } P(X) = 1/6 - \mathbf{B}_2(X) = \mathbf{B}_2(0) - \mathbf{B}_2(X).$$

Therefore the number s_χ can be written as

$$s_\chi = \sum_{a \in G} \chi(a) [\mathbf{B}_2(0) - \mathbf{B}_2(\langle a/p \rangle)],$$

and since χ is nontrivial, $\sum_{a \in G} \chi(a) = 0$, so that

$$s_\chi = -\sum_{a \in G} \chi(a) \mathbf{B}_2(\langle a/p \rangle).$$

Now the definition of the generalized Bernoulli numbers $B_{k,\chi}$ according to Leopoldt (see [L, page 37], or [W, Proposition 4.1]), gives

$$B_{k,\chi} = p^{k-1} \sum_{a=0}^{p-1} \chi(a) \mathbf{B}_k(\langle a/p \rangle);$$

hence we have

$$s_\chi = \frac{-1}{p} B_{2,\chi}$$

for an even nontrivial character χ on G.

Now the classical theorem that $B_{k,\chi}$ is nonzero when k and χ have the same parity shows that for every even character χ, $s_\chi \neq 0$, finishing the proof. This theorem is proved by using the associated L-series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

and noting first that for $n \geq 1$, $B_{n,\chi} = -nL(1-n, \chi)$ (see [I-R, Proposition 16.6.2] or [W, Theorem 4.2]). Hence it suffices to show that for even χ, $L(1-n,\chi)$ is nonzero. This follows from considering the functional equation for L, which for even χ can be written as (see [W, page 29])

$$L(1-s,\chi) = \Gamma(s)\frac{2}{\tau(\chi)}\left(\frac{p}{2\pi}\right)^{s} L(s,\overline{\chi})$$

where $\tau(\chi) = \sum_{a=1}^{p} \overline{\chi}(a)e^{2\pi ia/p}$ is a Gauss sum, and is known to be nonzero (it has absolute value $\sqrt{(p)}$, see [W, Lemma 4.8]). Therefore to finish we must check that for even $n \geq 2$, $L(n,\overline{\chi}) \neq 0$, and this follows since for any χ, $L(s,\chi)$ is nonzero whenever $\mathbf{Re}(s) > 1$, by considering the Euler product expansion for L, which is

$$L(s,\chi) = \prod_{p} (1-\chi(p)p^{-s})^{-1}.$$

□

The techniques of the proof given above was inspired by those used in a similar situation in [R, appendix to section 5], following [M-S]. I am indebted to Miles Reid for leading me in this direction.

The first corollary of the above Proposition is that the unoriented distribution numbers $M_{i,n}(S)$ for a torsion section S of prime order are determined by the linear system of Lemma 5.5. This proves that they are independent of S! Moreover, we can easily calculate them from the linear system. We require one lemma concerning the coefficient matrix $\mathbf{P}_p$:

Lemma 6.2. *For an odd prime p, and for any fixed $\alpha = 1,\dots,(p-1)/2$,*

$$\sum_{i=1}^{(p-1)/2} P(\langle \alpha i/p\rangle) = \frac{p^2-1}{12p}.$$

Proof. For fixed α, the integers αi as i ranges from 1 to $p-1$ themselves range over a complete set of representatives for the nonzero residue classes modulo p. Therefore, since $P(\langle \alpha i/p\rangle)$ is an even function of i, the set of values of this function for $i = 1,\dots,(p-1)/2$ are the same as the set of values for the function when $\alpha = 1$. In this range we have $\langle i/p\rangle = i/p$, so that for any α, the sum above is simply

$$\sum_{i=1}^{(p-1)/2} (i/p)(1-i/p) = \frac{p^2-1}{12p}$$

as claimed. □

Our main result is the following.

Theorem 6.3. *Let S be a torsion section of odd prime order p. Then*

$$M_{i,p}(S) = 2p/(p^2-1) \quad \text{if} \quad 1 \le i \le (p-1)/2,$$

and

$$M_{0,p}(S) = 1/(p+1).$$

If S is a torsion section of order 2, then

$$M_{1,2}(S) = 1/3 \quad \text{and} \quad M_{0,2}(S) = 2/3.$$

Proof. If p is odd, Lemma 6.2 shows that the matrix $\mathbf{P}_p$ has constant row sums. In other words, $\mathbf{P}_p\mathbf{1} = ((p^2-1)/12p)\mathbf{1}$, where $\mathbf{1}$ is the constant vector with every coordinate 1. Therefore a solution to the linear system $\mathbf{P}_p\mathbf{X} = (1/6)\mathbf{1}$ (of which the vector $\mathbf{M}_p$ of $M_{i,p}(S)$'s is a solution by Lemma 5.5) is given by a constant vector $(2p/(p^2-1))\mathbf{1}$. Since $\mathbf{P}_p$ is invertible, such a solution is unique, showing that for $i = 1, \ldots, (p-1)/2$, we must have $M_{i,p}(S) = 2p/(p^2-1)$. The value of $M_{0,p}$ is then gotten from (5.3).

For $p = 2$, the computations were made at the end of the last section. □

We will use the notation $M_{i,p}$ for $M_{i,p}(S)$ from now on. The computation of $M_{0,p}$ was done previously in [M-P3] using a quotient argument.

We view the above theorem as an "equidistribution" result: for a p-torsion section S, the number of times S meets a component which is "distance i" away from the zero component C_0, (measured in units of m_j/p), counted with multiplicity (m_j for the fiber F_j), is independent of i, for $i \neq 0$.

As a concrete example, take the elliptic $K3$ surface with 6 singular fibers, having $[m_j] = [1,1,1,7,7,7]$. Such a $K3$ surface exists, (see [M-P1]) and has a torsion section S of order 7 (see [M-P2]). The component numbers $\{k_j(S)\}$ must be $(0,0,0,a,b,c)$, where we can take the orientations of the I_7 fibers so that $0 \le a,b,c \le 3$. The equidistribution property above says that $M_{i,7} = 7/24$ for $i = 1,2,3$; hence we are forced to have $(k_j) = (0,0,0,1,2,3)$ after possibly reordering the fibers. Since $\chi(\mathcal{O}_X) = 2$, the sum of the component numbers $k_j(S)$ must be 6 by Corollary 4.3, so this is consistent. The quadratic relation for S in this case says that $1 \cdot 6/7 + 2 \cdot 5/7 + 3 \cdot 4/7 = 2\chi(\mathcal{O}_X) = 4$, which also checks out. Finally note that any multiple of S (except S_0 of course) has component numbers $\{0, 0, 0, 1 \text{ or } 6, 2 \text{ or } 5, 3 \text{ or } 4\}$ (after a possible permutation of the fibers) so that all of the relations above are satisfied for all multiples.

As another example, take the Beauville surface X_{6321}, which is a rational elliptic surface ($\chi(\mathcal{O}_X) = 1$) with four singular fibers of types I_6,

I_3, I_2, and I_1 (see [Be]). Such a surface has a finite Mordell-Weil group of order 6 (see [M-P4]). Using the relations given here for the component numbers, we can easily deduce them. Let S_2 be a torsion section of order 2 and S_3 be a torsion section of order 3. The component numbers for S_2 must be $(0 \text{ or } 3, 0, 0 \text{ or } 1, 0)$, and since their sum must be 4 by Corollary 4.1, we must have $(k_j(S_2)) = (3, 0, 1, 0)$. Note that these numbers do not change if we change orientations, since S_2 has order 2. The component numbers for S_3 must be $(0 \text{ or } 2 \text{ or } 4, 0 \text{ or } 1 \text{ or } 2, 0, 0)$, and we may choose orientations so that they are $(0 \text{ or } 2, 0 \text{ or } 1, 0, 0)$. By Corollary 4.3, these numbers must sum to 3, so that we must have $(k_j(S_3)) = (2, 1, 0, 0)$. In particular we have with these orientations that the sum of these sections, which is a section S_6 of order 6 generating the Mordell-Weil group of X, has component numbers $(5, 1, 1, 0)$. (We may re-orient the fibers to achieve $(k_j(S_6)) = (1, 1, 1, 0)$ if we desire.)

As a final example let us consider the Beauville surface $X = X_{3333}$, which is the Hessian pencil surface obtained by taking a smooth cubic curve C in the plane and considering the pencil formed by C and its Hessian H. One obtains after blowing up the 9 base points of this pencil (which are the 9 flexes of C) a rational elliptic surface ($\chi = 1$) with four I_3 fibers. (One gets the same surface starting from any smooth C!). The Mordell-Weil group of X is of order 9, with the zero-section S_0 and 8 other sections of order 3. The component numbers for a section are all 0, 1, or 2, and if S is not the zero section, then the component number sum formula Corollary 4.4 says that exactly one $k_j(S)$ is 0, all others are non-zero. We may choose an ordering of the fibers and orientations so that one of the order 3 sections S_1 has component numbers $(0, 1, 1, 1)$. Let S_2 be another order 3 section, not equal to $2S_1$, so that S_1 and S_2 generate the Mordell-Weil group. By reordering the fibers we may assume that the component numbers for S_2 are $(a, 0, b, c)$ where a, b, c are not 0; after reorienting the first fiber we may further assume that $a = 1$. Since the sum $S_3 = S_1 + S_2$ will then have component numbers $(1, 1, b+1, c+1)$ and must have three non-zero component numbers, we may (after possibly switching the last two fibers) assume that $b = 2$ and $c = 1$, so that $(k_j(S_2)) = (1, 0, 2, 1)$. This then determines the component numbers for all 8 nonzero sections of X.

As an application of these computations of the unoriented distribution numbers $M_{i,p}$ for prime p, we obtain a divisibility result. Since each of the numbers $12\chi M_{i,p}(S)$ is an integer, and since p and $p^2 - 1$ are always relatively prime, we obtain the following.

Corollary 6.4. *Let p be an odd prime and suppose that f admits a section of order exactly p. Then $(p^2-1)/2$ divides $e = 12\chi$.*

This improves a divisibility result stated in [M-P3, page 255] that $p+1$ divides $e = 12\chi$, which was obtained by using $M_{0,p}$ only. Moreover this is sharp; for the elliptic modular surface (associated to the group $\Gamma_1(p)$) which has a section of order p, the Euler number is exactly $e = (p^2 - 1)/2$ (see [Shm] or [C-P]). In some sense we can view this result as enabling us to "divine" the Euler number of such a universal surface, without constructing it: if it existed, it surely would have minimal e.

We also have recovered what Persson and I referred to as the "Fixed Point Rule" in [M-P1, Corollary 5.5] (see also [M-P2, Lemma 3(c)] and [M, Corollaries X.3.3 and X.3.4]). Again these works concentrated on the K3 surface case, which has $\chi(\mathcal{O}_X) = 2$; the statement in this case was that if S is a p-torsion section on a K3 elliptic surface, with component numbers $\{k_j\}$, then

$$\sum\{m_j \ : \ k_j \neq 0\} = 24p/(p+1).$$

In general of course we have (using the results of this article) that

$$\begin{aligned} \sum\{m_j \ : \ k_j \neq 0\} &= 12\chi - \sum\{m_j \ : \ k_j = 0\} \\ &= 12\chi - 12\chi \cdot M_{0,p}(S) \\ &= 12\chi(1 - 1/(p+1)) \\ &= 12\chi p/p + 1, \end{aligned}$$

which generalizes the Fixed Point Rule as stated in the above articles.

Finally, I would like to mention a "dual" equidistribution property. It concerns not the indices of the components other than the zero component which a p-torsion section S hits, but rather the position in $C_0^{(j)}$ which S hits, when it does hit this zero component. Specifically, each $C_0^{(j)}$ can be identified with $\mathbb{C}^*$ by sending the nodes to 0 and ∞, and sending the point $S_0 \cap C_0^{(j)}$ to 1. (Note that if one exchanges the nodes, the identification with $\mathbb{C}^*$ changes by the inversion $z \mapsto 1/z$.) Any torsion section S of order p will then hit $C_0^{(j)}$ in a p^{th} root of unity. The equidistribution property in this context is that each p^{th} root of unity occurs equally often, (counted properly, and up to the inversion given by exchanging the nodes).

I do not know how to detect this root of unity using the elementary types of techniques used in this paper. Of course this property, properly formulated, is invariant under base change, and so it would suffice to check it for the modular surfaces. Peter Stiller and I have done this and this work will appear in a forthcoming paper.

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