The Segre and Harbourne-Hirschowitz Conjectures

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Abstract

In this paper we prove the equivalence of two conjectures, one by B. Segre, the other by Harbourne and Hirschowitz, concerning linear systems of plane curves with general assigned base points of given multiplicities. Then we deduce, under the assumption that either one of the two conjectures is true, some general results about the structure of the aforementioned linear systems.

1 Introduction

Consider the linear system of plane curves of a given degree d having at n specified, but general, points of the plane, singularities of multiplicities say $m_1, ..., m_n$. The classical multivariate polynomial interpolation problem consists in evaluting the dimension of the linear system in question. Since the conditions one imposes, by imposing the n multiple points, are $\sum_{i=1}^n \frac{m_i(m_i+1)}{2}$ linear and since the plane curves of degree d depend on $\frac{d(d+3)}{2}$ parameters, i.e., up to a scalar, the coefficient of their equation, we see that the expected dimension of the linear system we are dealing with is the maximum between $\frac{d(d+3)}{2} - \sum_{i=1}^n \frac{m_i(m_i+1)}{2}$ and -1, the latter being the dimension of the empty system. A system for which the dimension is bigger than the expected one is called *special*, and it is a long-standing problem to classify all special linear systems of the above type.

One way speciality of linear systems can arise, is if they have as a multiple base curve a so called (-1)-*curve*, i.e. a rational curve which, on the blow up of the plane at the multiple base points, ends up having self-intersection -1. An important conjecture due to Harbourne [Ha1] and Hirschowitz [Hi] (see §3) says that this is *the only way* speciality can arise. Some authors, among them Gimigliano [Gi], also pointed out an earlier conjecture, formulated by B. Segre [Se], to the effect that if a system is special then it has some multiple curve in its base locus (see §2). Clearly Harbourne-Hirschowitz conjecture implies Segre's one. In this paper we prove that also the converse holds (see §4). This is also implied by some results of Nagata's [Na], but the proof presented here, which relies only on basics of surface theory, is really very elementary and faster than Nagata's. We also classify, up to Cremona transformations and under the assumption that one of the conjectures holds, all systems as above whose general curve is reducible (see §5) as well as all special system for which the multiplicity of the assigned base points are the same, i.e. the so called *homogeneous* linear system see §6). Finally, in §7, we observe that Segre's conjecture implies a famous conjecture of Nagata also concerning plane curves.

The multivariate polynomial interpolation problem has, of course, relations and connections with numerical analysis. For this reason some of these issues have been addressed by numerical analysts too. For general references we refer the reader to the expository article [C].

2 The Segre Conjecture

Let $p_1, ..., p_n$ be general points in the complex projective plane $\mathbb{P}^2$ and let $m_1, ..., m_n$ be positive integers. We let $\mathcal{L}_d$ be the linear system of plane curves of degree d and $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ the sub-linear system of $\mathcal{L}_d$ of curves having multiplicity at least m_i at the point p_i, $i = 1, ..., n$. If $m_i = 1$ we suppress the superscript m_i for p_i in $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$.

Let $\pi : S \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points $p_1, ..., p_n$. Let $\mathcal{L}$ be a line bundle on S, or, by abusing notation, the corresponding complete linear system. One defines the *virtual dimension* of $\mathcal{L}$ to be:

$$\nu(\mathcal{L}) := \chi(\mathcal{L}) - 1 = \frac{\mathcal{L} \cdot (\mathcal{L} - K_S)}{2}$$

where K_S is the canonical class on S.

If C is any divisor on S, we similarly define $\nu(C) := \chi(\mathcal{O}_S(C)) - 1$. The Riemann-Roch Theorem says that if $\mathcal{L}$ is effective, then

$$\dim(\mathcal{L}) = \nu(\mathcal{L}) + h^1(S, \mathcal{L}) \tag{2.1}$$

since $h^2(\mathcal{L}) = 0$. One also defines the *expected dimension* of $\mathcal{L}$ to be

$$\epsilon(\mathcal{L}) := \max\{\nu(\mathcal{L}), -1\}.$$

If C is any divisor on S we can accordingly define $\epsilon(C) := \max\{\nu(C), -1\}$.

One says that a linear system $\mathcal{L}$ on S is *non-special* if its dimension equals the expected dimension. This is equivalent to saying that $\mathcal{L}$ is non-special if and only if $h^0(S, \mathcal{L}) \cdot h^1(S, \mathcal{L}) = 0$. In other words $\mathcal{L}$ is non special if and only if either it is empty or it is *regular*, namely not empty and with $h^1(S, \mathcal{L}) = 0$.

Let now H be the pull-back via π of a general line of the plane and let $E_1, ..., E_n$ be the exceptional divisors contracted by π to the points $p_1, ..., p_n$. The *proper transform* of $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ on S is the complete linear system $\mathcal{L} := |dH - m_1E_1 - ... - m_nE_n|$. By abusing notation, we will denote by $\mathcal{L}$ also the line bundle associated to this linear system.

We apply the language of virtual and expected dimension to the system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ on the plane also, by using the corresponding notions of the proper transform. In particular, the *virtual dimension* of $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ is

$$\nu(\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})) := \nu(\mathcal{L}) = \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$$

and the *expected dimension* of $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ is

$$\epsilon(\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})) := \epsilon(\mathcal{L}).$$

One says that a system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ of plane curves is *non-special* if the proper transform $\mathcal{L}$ on S is such.

A linear system $\mathcal{L}$ on S, which is not empty, is called *reducible* [resp. *reduced*] if its general curve C is reducible [resp. reduced]. Bertini's theorem tells us that, if $\mathcal{L}$ is reducible, then either it has some fixed components or it is composed with a rational pencil $\mathcal{P}$, i.e. the movable part of $\mathcal{L}$ consists of the sum of $h \geq 2$ curves of $\mathcal{P}$. The following conjecture is due to B. Segre (see [Se]):

Conjecture 2.2 (Segre's Conjecture) *Suppose that $\mathcal{L}$ as above is nonempty and reduced. Then $\mathcal{L}$ is non-special.*

Since a plane curve is reduced if and only if it has isolated singularities, another way of phrasing Segre's Conjecture is: if the general member of $\mathcal{L}$ has isolated singularities, then $H^1(S, \mathcal{L}) = 0$. In this form it may generalize to higher dimensions (see [C]).

3 The Harbourne-Hirschowitz Conjecture

Next we need some more definitions. A smooth irreducible rational curve E on S such that $E^2 = -1$ is called a (-1)-*curve.*

Let C be the general curve of the linear system $\mathcal{L}$. The linear system $\mathcal{L}$ is said to be (-1)-*reducible*, if $\mathcal{L}$ is not empty and there is some (-1)-curve E such that $E \cdot C < 0$. In this case E is a fixed component of $\mathcal{L}$.

A linear system $\mathcal{L}$ on S, which is not empty, is called (-1)-*special*, if there is some (-1)-curve E such that $E \cdot C < -1$, where, again, C is the general curve of $\mathcal{L}$. If $E \cdot C = -h, h > 0$, then hE appears in the fixed part of $\mathcal{L}$. Moreover $E \cdot (C - hE) = 0$ and $h^1(S, \mathcal{O}_S(C)) \geq \binom{h-1}{2}$ (see [CM1]). This motivates the above definition, inasmuch as a (-1)-special system is indeed special.

Similar definitions can be given for a linear system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ of plane curves, inheriting them from the proper transform system $\mathcal{L}$ on S. In particular we will speak of the *self-intersection* of the curves of the linear system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ referring to the self-intersection of the curves in the proper transform system $\mathcal{L}$ on S, etc.

The following conjecture is essentially due to Harbourne and Hirschowitz (see [Ha1], [Hi]) and is related to an earlier conjecture by Gimigliano (see [Gi]). We will formulate it as follows.

Conjecture 3.1 *In the above situation one has:*

(i) a linear system $\mathcal{L}$ on S is special if and only if it is (-1)-special;

(ii) if a linear system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ is non-empty then the general curve in it has singular points of multiplicities exactly $m_1, ..., m_n$ at $p_1, ..., p_n$;

(iii) if a linear system $\mathcal{L}$ on S is non-empty and non-special, then it is irreducible unless either it is (-1)-reducible or it consists of a unique multiple irreducible curve of genus 1 and self-intersection 0 or it is composed of a pencil of rational curves with self-intersection zero;

(iv) if a linear system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ is non-empty and non-special, and if the general curve in the system is irreducible, then it has ordinary singular points of multiplicities $m_1, ..., m_n$ at $p_1, ..., p_n$ and it has no other singularities.

Notice that parts (i) and (iii) of the conjecture in particular apply to systems $\mathcal{L}$ corresponding to systems $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ of plane curves. Hence, if the conjecture is true, then a system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ is special if and only if it is (-1)-special, etc.

4 Segre Implies Harbourne-Hirschowitz

Note that the Harbourne-Hirschowitz Conjecture immediately implies Segre's Conjecture, since (-1)-special systems are not reduced. In this section we will show that in fact Segre's Conjecture is not weaker that the Harbourne-Hirschowitz.

We keep the notation from the previous section. We will mainly work with linear systems on the blown-up surface S rather than with systems $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$. First we notice that Segre's Conjecture implies the following:

Theorem 4.1 *Suppose that Segre's Conjecture is true. Let $p_1, ..., p_n$ be general points in the plane and let S be the blow-up of the plane at $p_1, ..., p_n$. Let $\mathcal{L}$ be a complete linear system on S which is not empty and such that its general curve C is irreducible and reduced of arithmetic genus g. Then:*

(i) $H^1(S, \mathcal{O}_S(C)) = 0$, $\dim(\mathcal{L}) = \nu(\mathcal{L})$, and $\mathcal{L}$ is non-special;

(ii) $C^2 \geq g - 1$ and therefore $C^2 < 0$ if and only if C is a (-1)-curve.

Proof: The assertion about $H^1(S, \mathcal{O}_S(C))$ is exactly Segre's Conjecture; the rest of (i) follows from (2.1). Part (ii) follows by Riemann-Roch and $\nu(C) \geq 0$: we have $C^2 + C \cdot K_S = 2g - 2$, and $2\nu(C) = C^2 - C \cdot K_S \geq 0$; summing these gives the result. $\diamondsuit$

¿From this theorem we will deduce the following result, which relates to parts (i), (ii) and (iii) of the Harbourne-Hirschowitz Conjecture.

Theorem 4.2 *Suppose that Segre's Conjecture is true. Let $p_1, ..., p_n$ be general points in the plane and let S be the blow-up of the plane at $p_1, ..., p_n$. Let $\mathcal{L}$ be a complete, nonempty linear system on S. Then:*

(i) If $\mathcal{L}$ is special then $\mathcal{L}$ is (-1)-special.

(ii) If $\mathcal{L}$ is non-special, and the general element of $\mathcal{L}$ is not reduced, then $\dim(\mathcal{L}) = 0$ and $\mathcal{L}$ consists of a unique curve of the form $mC + D_1 + ... + D_h$, where C is an elliptic curve with self intersection 0 and $D_1, ..., D_h$ are (-1)-curves which are mutually disjoint and disjoint from C.

(iii) If $\mathcal{L}$ is non-special, the general curve in $\mathcal{L}$ is reduced, and there are fixed components of $\mathcal{L}$, then the fixed components of $\mathcal{L}$ are disjoint curves which are (-1)-curves, plus, at most one more curve of positive genus. If there is such a fixed component of positive genus, then $\mathcal{L}$ has dimension 0.

(iv) *If $\mathcal{L}$ is non-special of positive dimension, then its general curve is reduced and its movable part is disjoint from the fixed part, consisting of a union of disjoint (-1)-curves. The movable part, if reducible, is composed of a base point free pencil of rational curves.*

We will prove Theorem (4.2) in several steps. We start with the following lemma:

Lemma 4.3 *Suppose that Segre's Conjecture is true. Let $p_1, ..., p_n$ be general points in the plane and let S be the blow-up of the plane at $p_1, ..., p_n$. Suppose that C and D are general members of their linear systems $|C|$ and $|D|$ on S respectively, and that they are both irreducible and disjoint. If C has positive arithmetic genus, then D has arithmetic genus zero and* $\dim |D| = 0$.

Proof: Consider the exact sequence

$$0 \to \mathcal{O}_S(D - C) \to \mathcal{O}_S(D) \to \mathcal{O}_C(D) \cong \mathcal{O}_C \to 0$$

the final isomorphism holding since C and D are disjoint. By Theorem (4.1,i), the higher cohomology of $\mathcal{O}_S(D)$ is zero, so that $h^2(\mathcal{O}_S(D - C)) = h^1(\mathcal{O}_C) \geq 1$. Hence by Serre duality $|K_S + C - D|$ is not empty. Reversing the roles of C and D we see by symmetry that $h^2(\mathcal{O}_S(C - D)) = h^1(\mathcal{O}_D)$. If D has positive genus, then this is also nonzero; hence $|K_S + D - C|$ is not empty, and we conclude that $|2K_S|$ is not empty, a contradiction. Therefore D has genus zero.

If $\dim |D| \geq 1$, then $h^0(\mathcal{O}_S(D)) \geq 2$, so the above exact sequence implies that $h^0(\mathcal{O}_S(D - C)) \geq 1$, i.e., $|D - C|$ is not empty. Since we have seen that $|K_S + C - D|$ is also not empty, we conclude that $|K_S|$ is not empty, which is a contradiction. Therefore $\dim |D| = 0$ as claimed. ◇

The following lemma analyzes the fixed components of a linear system $\mathcal{L}$ on a general blowup S of the plane.

Lemma 4.4 *Suppose that Segre's Conjecture is true. Let S and $\mathcal{L}$ be as above, with $\mathcal{L}$ not empty, and let C be the general curve in $\mathcal{L}$. Let us write:*

$$C = h_1C_1 + \ldots + h_kC_k + D_1 + \ldots + D_h + M, \quad k \geq 0, \quad h \geq 0$$

where $C_1, \ldots, C_k, D_1, \ldots, D_h$ are the distinct, irreducible, fixed components of $\mathcal{L}$, and M is the general curve in the movable part of the system (which may be empty). Also we assume $h_1 \geq ... \geq h_k \geq 2$. Then one has:

(i) $C_i^2 = C_i \cdot K_S$, $C_i^2 \leq 0$, $i = 1, \ldots, k$; $D_j^2 = D_j \cdot K_S$, $j = 1, \ldots, h$;

(ii) $C_i \cdot C_j = 0$, $1 \leq i < j \leq k$; $D_i \cdot D_j = 0$, $1 \leq i < j \leq h$; $C_i \cdot D_j = 0$, $i = 1, \ldots, k, j = 1, \ldots, h$;

(iii) *if F is any irreducible component of M, then* $F \cdot C_i = F \cdot D_j = 0$, $i = 1, \ldots, k$, $j = 1, \ldots, h$.

Proof: One has

$$\dim|C_i| = \dim|D_j| = 0, \text{ for } i = 1, \dots, k, \;\; j = 1, \dots, h$$

since they are fixed components. By Theorem (4.1), this implies that:

$$\nu(C_i) = \nu(D_j) = 0, \text{ for } i = 1, \dots, k, \;\; j = 1, \dots, h$$

i.e.

$$C_i^2 = C_i \cdot K_S, \text{ for } i = 1, \dots, k; \quad D_j^2 = D_j \cdot K_S, \text{ for } j = 1, \dots, h.$$

Also

$$\dim|C_i + C_j| = 0, \text{ for } 1 \leq i < j \leq k,$$

and hence (using (2.1)) we have

$$\nu(C_i + C_j) \leq 0, \text{ for } 1 \leq i < j \leq k.$$

This yields:

$$C_i \cdot C_j = 0, 1 \leq i < j \leq k$$

and, in a completely similar manner, we prove the rest of (ii).

Let F be an irreducible component of M. Since F must move in its linear system, in this case we have

$$\dim|F| = \nu(F) > 0.$$

Since $\dim|F + A| = \dim|F|$ for $A = C_i$ or $A = D_j$, we must have

$$\nu(F + C_i) \leq \nu(F), \text{ for } i = 1, \dots, k \text{ and } \nu(F + D_j) \leq \nu(F), \text{ for } j = 1, \dots, h$$

which yields (iii), since for any curve A we have

$$2(\nu(F + A) - \nu(F)) = 2A \cdot F + A^2 - A \cdot K_S$$

and applying this to either $A = C_i$ or $A = D_j$ and using the proved part of (i) gives $F \cdot C_i \leq 0$ and $F \cdot D_j \leq 0$. Since F, C_i and D_j are actual curves on S, we must have $F \cdot C_i = F \cdot D_j = 0$.

Finally we have:

$$\dim|2C_i| = 0, \text{ for } i = 1, \dots, k,$$

hence

$$\nu(2C_i) \leq 0, \text{ for } i = 1, \dots, k.$$

This reads:

$$2C_i^2 \leq C_i \cdot K_S, \text{ for } i = 1, \dots, k,$$

which, together with $C_i^2 = C_i \cdot K_S$ forces $C_i^2 \leq 0$. This finishes the proof of (i). $\diamondsuit$

Next we analyze the nonreduced systems, and prove the:

Proposition 4.5 *Suppose that Segre's Conjecture is true. With the same assumptions and notation as in Lemma (4.4), assume that the general member of the system is nonreduced, i.e., that $k \geq 1$. Then either one of the curves C_i is a (-1)-curve, in which case the system is (-1)-special, or $k = 1$, $C = C_1$ is elliptic, $D_1, \ldots, D_h$ are (-1)-curves, $M = 0$ and the system is non-special.*

Proof: By the adjunction formula, we have $2C_i^2 = C_i^2 + C_i \cdot K_S \geq -2$ for $i = 1, \ldots, k$. Hence, if $C_i^2 < 0$, then one has $C_i^2 = C_i \cdot K_S = -1$ and C_i is a (-1)-curve. Since $h_i \geq 2$ the system is special in this case.

Suppose that $C_i^2 = 0$ for all $i = 1, \ldots, k$, i.e. each C_i is an elliptic curve. If $k \geq 2$ we have a contradiction, applying Lemma (4.3), since the C_i's are disjoint. Therefore we must have $k = 1$ and we set $C_1 = C$, $h_1 = h$.

Let F be an irreducible component of M; since $F \cdot C = 0$, we apply Lemma (4.3) again and conclude that F has genus zero and $\dim |F| = 0$. This gives a contradiction, since F moves in its linear system. We conclude that $M = 0$.

Finally for every $j = 1, \ldots, h$, we apply Lemma (4.3) to C and D_j and conclude that each D_j is rational. Then Lemma (4.4,i) implies that it is a (-1)-curve (since $D_j^2 = D_j \cdot K_S$ and the sum is -2 by rationality).

The assertion about the non-speciality of the system is an easy computation: $\nu(h_1 C_1 + D_1 + \ldots + D_h) = 0$ and the system also has dimension zero. $\diamondsuit$

We next analyze systems with no fixed components:

Proposition 4.6 *Suppose that Segre's Conjecture is true. With the same assumptions and notation as in Lemma (4.4), assume that $k = h = 0$, i.e. $\mathcal{L}$ has no fixed components. Then either M is irreducible or $\mathcal{L}$ is composed of a base point free pencil of rational curves; in either case the system is non-special.*

Proof: If M is irreducible, then the system is non-special by Theorem (4.1,i). Assume M is reducible. Then $\mathcal{L}$ is composed of a pencil, namely $M \equiv \mu F$, where $|F|$ is a pencil and $\mu \geq 2$. By Theorem (4.1) we have

$$\dim |F| = \nu(F) = 1,$$

i.e. $F^2 - F \cdot K_S = 2$. On the other hand one has

$$\nu(\mu F) \leq \dim |\mu F| = \dim |M| = \mu$$

which gives $\mu^2 F^2 - \mu F \cdot K \leq 2\mu$, or $\mu F^2 - F \cdot K \leq 2$. Subtracting $F^2 - F \cdot K_S = 2$ from this inequality gives $(\mu - 1)F^2 \leq 0$. Since F moves in a pencil, we must have $F^2 \geq 0$, which forces $F^2 = 0$; then $F \cdot K_S = -2$, which implies that $|F|$ is a base-point-free rational pencil as claimed. Moreover in this case then $\nu(\mu F) = \mu$, so that $M \equiv \mu F$ is non-special also. $\diamondsuit$

Now we can analyse the reduced linear systems with fixed components and prove the following:

Proposition 4.7 *Suppose that Segre's Conjecture is true. With the same assumptions and notation as in Lemma (4.4), assume that $k = 0$ and $h > 0$. Then either all the curves D_j are (-1)-curves or all of them, but one, say D_1, are (-1)-curves. In the latter case $M = 0$ and in both cases the system is non-special.*

Proof: The proof imitates the one of Proposition (4.5). Since the D_j's are disjoint, we may apply Lemma (4.3) and conclude that at most one of them is non-rational. Any rational D_j must be a (-1)-curve (see Lemma (4.4)).

Suppose that one of them, say D_1, is not rational and consider an irreducible component F of M. Since $F \cdot D_1 = 0$, we apply Lemma (4.3) and conclude that F is rational and does not move in its linear system, a contradiction. Therefore $M = 0$.

If $M = 0$, the assertion about the non-speciality of the system is a straightforward computation: $\nu(D_1 + \cdots + D_h) = 0$, and the dimension is also zero. If $M \neq 0$, then each D_j is a (-1)-curve. Again one checks that

$$\nu(D_1 + \cdots + D_h + M) = \nu(M)$$

so that since these systems have the same dimension, $D_1 + \cdots + D_h + M$ is nonspecial if and only if M is nonspecial. The result now follows from the previous Proposition. $\diamondsuit$

The above Propositions prove Theorem (4.2). Next we see how Conjecture (3.1, i, ii, iii) follows from the above results.

Theorem 4.8 *Suppose that Segre's Conjecture is true. Then parts (i), (ii) and (iii) of Conjecture (3.1) hold.*

Proof: Part (i) of the Conjecture is nothing but Theorem (4.2, i). Part (iii) also follows from Theorem (4.2, ii, iii, iv).

As for part (ii), let $\mathcal{L}$ be the proper transform of $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ on S. Let C be the general curve in $\mathcal{L}$. Suppose that the exceptional divisor E_i contracted to p_i is an irreducible component of multiplicity $h \geq 1$ of C. Then, by Lemma (4.4, ii, iii) one has $E \cdot (C - hE) = 0$ hence $m_i = C \cdot E = hE^2 = -h < 0$, a contradiction. Hence E_i is not a component of C. Since $C \cdot E_i = m_i$, then (3.1, ii) also follows. $\diamondsuit$

5 Reducible, Non-special Linear Systems

In this section we will describe non-special, reducible linear systems which we met in Conjecture (3.1, iii) and in Theorem (4.2). First a remark is in order. We express it in the following Lemma which is a corollary of Theorem (4.1).

Lemma 5.1 *Suppose that Segre's Conjecture is true. Let $p_1, ..., p_n$ be general points in the plane and let S be the blow-up of the plane at $p_1, \ldots, p_n$. Let X be a minimal rational surface obtained from S by a sequence of contractions of (-1)-curves. Then either $X = \mathbb{P}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$. In the former case the map $\sigma : S \to X = \mathbb{P}^2$ is a sequence of contractions of n disjoint (-1)-curves of S to general points $q_1, \ldots, q_n$ of the plane; in the latter case the map $\sigma : S \to X = \mathbb{P}^1 \times \mathbb{P}^1$ is a sequence of contractions of $n-1$ disjoint (-1)-curves of S to general points $r_1, \ldots, r_{n-1}$ of $\mathbb{P}^1 \times \mathbb{P}^1$. In the latter case S also contracts to $\mathbb{P}^2$, as in the former case.*

Proof: The surface X is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$, with either $n \geq 2$ or $n = 0$. But $\mathbb{F}_n$ contains a smooth rational curve with self-intersection $-n$; in this case Theorem (4.1,ii) implies that $n = 0$. Hence $X = \mathbb{P}^2$ or $X = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Suppose $X = \mathbb{P}^2$. The map $\sigma : S \to X = \mathbb{P}^2$ is a sequence of n contractions of (-1)-curves $\sigma_i : S_i \to S_{i+1}$, $i = 0, \ldots, n$, where $S_0 = S$ and $S_n = \mathbb{P}^2$. Clearly, each surface S_i shares with S the property of not containing any irreducible curve of negative self-intersection, except for (-1)-curves. One moment of reflection shows that this implies that σ is obtained by contracting n disjoint (-1)-curves of S to points $q_1, ..., q_n$ of the plane. The fact that $q_1, ..., q_n$ must be general follows by an obvious parameter count.

A similar argument works in the case $X = \mathbb{P}^1 \times \mathbb{P}^1$.

Finally the last assertion follows from noting that one blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ contracts to $\mathbb{P}^2$. $\diamondsuit$

With the above notation, suppose $X = \mathbb{P}^2$ (which we may assume by the last assertion). Let $\sigma : S \to X$ be a birational map obtained by a sequence of contractions of n disjoint (-1)-curves of S as in the statement of Lemma (5.1) above. Then $\sigma \circ \pi^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ is a *Cremona transformation.*

We will use such Cremona transformations to understand the systems $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ which are non-special but reducible. Given a non-special system $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$, we consider its proper transform $\mathcal{L}$ on S. Then $\sigma_*(\mathcal{L})$ is a linear system of the type $\mathcal{L}_\delta(q_1^{n_1}, \ldots, q_r^{n_r})$, where $q_1, \ldots, q_r$ are general points in $\mathbb{P}^2$, and actually $r \leq n$. We will define this system to be the *Cremona transform* of $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ via the given Cremona transformation $\sigma \circ \pi^{-1}$. Of course the dimension of the two systems are the same.

If $\mathcal{L}'$ is the system associated to $\mathcal{L}_\delta(q_1^{n_1}, \ldots, q_r^{n_r})$ on S, it is clear that $\mathcal{L} \otimes \mathcal{L}'^{-1} \cong \mathcal{O}_S(E)$ where $E = E_1 + ... + E_h$ and $E_1, \ldots, E_h$ are the distinct (-1)-curves contracted by σ which are in the base locus of $\mathcal{L}$. Hence also $\nu(\mathcal{L}) = \nu(\mathcal{L}')$.

Having all this in mind, we can state our next theorem:

Theorem 5.2 *Suppose that Segre's Conjecture is true. Let $p_1, \ldots, p_n$ be general points in the plane and let S be the blow-up of the plane at $p_1, \ldots, p_n$. Let $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ be a non-special, reducible linear system. Then it can be Cremona transformed either to:*

(i) the linear system $\mathcal{L}_d(p^d)$ whose general member consist of d lines through a point p, or;

(ii) the linear system $\mathcal{L}_{3m}(q_1^m, \ldots, q_9^m)$ which consists of the unique cubic curve C containing the nine general points $q_1, \ldots, q_9$, counted with multiplicity m.

The main tool for proving this result is the following classical result, apparently already known to M. Noether:

Theorem 5.3 *Let $p_1, \ldots, p_n$ be general points in the plane and let $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ be a linear system with $m_1 \geq m_2 \geq \ldots \geq m_n$ and $m_3 \geq 1$. Let S be the blow-up of the plane at $p_1, \ldots, p_n$ and let $\mathcal{L}$ be the proper transform of $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ on S. Assume that the general member C of $\mathcal{L}$ is irreducible and let g be its arithmetic genus. If $(m_3 - 1)C^2 \geq 2m_3(g - 1)$ and either the strict inequality holds or $m_3 < m_1$, then $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ can be Cremona transformed to a system $\mathcal{L}_\delta(q_1^{n_1}, \ldots, q_r^{n_r})$ with $\delta < d$.*

If $(m_3 - 1)C^2 = 2m_3(g - 1)$ and $m_1 = m_2 = m_3 = m$, then $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ can be Cremona transformed to a system $\mathcal{L}_\delta(q_1^{n_1}, \ldots, q_r^{n_r})$ with $\delta < d$ unless $d = 3m$.

Proof: We want to prove that $d < m_1 + m_2 + m_3$, since then the quadratic transformation based at p_1, p_2, p_3 does the job of Cremona transforming the given linear system into a new one with lower degree.

One has:

$$d(d-3) - \sum_{i=1}^{n} m_i(m_i - 1) = 2g - 2 \quad \text{and} \quad d^2 - \sum_{i=1}^{n} m_i^2 = C^2.$$

Multiplying the first by m_3 and the second by $1 - m_3$, summing, and re-arranging, yields

$$\begin{aligned} d(3m_3 - d) + m_1^2 + m_2^2 - m_3(m_1 + m_2) \quad & - \quad [(m_3 - 1)C^2 - 2m_3(g-1)] \\ & = m_3 \textstyle\sum_{i=4}^{n} m_i - \sum_{i=4}^{n} m_i^2 \geq 0. \end{aligned}$$

The term $d(3m_3 - d)$ is maximized for $d = 3m_3/2$, and monotonically decreases for larger d. Hence if $d \geq m_1 + m_2 + m_3$ we would thus have

$$\begin{aligned} (m_1 + m_2 + m_3)(2m_3 - m_1 - m_2) \quad & + \quad m_1^2 + m_2^2 - m_3(m_1 + m_2) \\ & - \quad [(m_3 - 1)C^2 - 2m_3(g-1)] \geq 0 \end{aligned}$$

which simplifies to the inequality

$$2(m_3^2 - m_1 m_2) - [(m_3 - 1)C^2 - m_3(2g - 2)] \geq 0.$$

Since $m_1 \geq m_2 \geq m_3$, the first term here is non-positive, and is strictly negative if $m_3 < m_1$. Therefore by the first assumption we have a contradiction.

The last assertion is also clear. $\diamondsuit$

Corollary 5.4 *Suppose that Segre's Conjecture is true. Let $p_1, \ldots, p_n$ be general points in the plane. An irreducible linear system $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ whose general member has genus $g \leq 1$ can be Cremona transformed to one of the following types:*

- *a linear system of lines, dimension $l \leq 2$ and $g = 0$*
- *the linear system of all conics, dimension 5 and $g = 0$*
- *a linear system $\mathcal{L}_d(p^{d-1})$, $d \geq 2$, dimension $2d$ and $g = 0$*
- *a linear system $\mathcal{L}_d(p^{d-1}, q)$, $d \geq 2$, dimension $2d - 1$ and $g = 0$*
- *a linear system of cubics $\mathcal{L}_3(p_1, \ldots, p_h)$, $h \leq 9$, dimension $9 - h$ and $g = 1$*
- *a linear system $\mathcal{L}_4(p_1^2, p_2^2)$, dimension 8 and $g = 1$.*

Proof: Assume $g = 0$. Then Theorem (5.3) tells us that $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ can be Cremona transformed to a system of lower degree, unless $n \leq 2$. In other words $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ can be Cremona transformed to a system of one of the forms $\mathcal{L}_\delta$, $\mathcal{L}_\delta(q^n)$, or $\mathcal{L}_\delta(q_1^{n_1}, q_2^{n_2})$. It is then easy to see that the only possible cases are those listed in our statement.

Let $g = 1$. Theorem (5.3) tells us that $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ can be Cremona transformed to a system of lower degree, unless either $n \leq 2$, or $m_1 = m_2 = m_3 = m$, $C^2(m_3 - 1) = 0$ and $d = 3m$. Therefore the possibilities are:

(a) $n \leq 2$;

(b) $m_1 = m_2 = m_3 = 1$;

(c) $m_1 = m_2 = m_3 = m \geq 2$, $C^2 = 0$, $d = 3m$.

Cases (a) and (b) easily lead to the possibilities we listed in our statement. Case (c) cannot occur. In fact $d = 3m$ implies $n = 9$ and $m_1 = \ldots = m_9 = m$. Since there is a unique cubic E through $p_1, ..., p_9$ and since, by hypothesis, the original system contains an irreducible curve C, we see that C should vary in a pencil, the one generated by C and mE. Then we would have $\mathcal{O}_C(C) \simeq \mathcal{O}_C$, hence $h^1(C, \mathcal{O}_C(C)) = 1$ contradicting Theorem (4.1, i). ♢

Remark: The numerical case (c) which appears in the analysis of the case $g = 1$ in the above proof is impossible if $p_1, \ldots, p_9$ are general points, but can occur for special positions of the points, leading to the well known *Halphen pencils* (see [CD]).

We are now in a position to give the:

Proof: [of Theorem (5.2)]. Let $\mathcal{L}$ be, as usual, the proper transform of $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ on S. By blowing-down all the (-1)-curves that appear in the fixed part of $\mathcal{L}$, we may reduce ourselves, up to Cremona transformations, to assume that $\mathcal{L}$ is not (-1)-reducible. Then either $\mathcal{L}$ consists of a unique multiple elliptic curve with self-intersection 0, or it is composed of a pencil of rational curves with self intersection 0. The conclusion then follows by Corollary (5.4). ♢

6 Homogeneous Linear Systems

We want to address now the analysis of the so-called *homogeneous linear systems*, i.e. linear systems $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ with $m_1 = \ldots = m_n = m$. We will denote such systems briefly by $\mathcal{L}_d(n^m)$.

Our result is the following:

Theorem 6.1 *Suppose that Segre's Conjecture is true. One has:*

(i) a homogeneous linear system $\mathcal{L}_d(n^m)$ is special if and only if one of the following cases occurs:

1. $n = 2, m \leq d \leq 2m - 2$
2. $n = 3, \frac{3m}{2} \leq d \leq 2m - 2$
3. $n = 5, 2m \leq d \leq \frac{5m-2}{2}$
4. $n = 6, \frac{12m}{5} \leq d \leq \frac{5m-2}{2}$
5. $n = 7, \frac{21m}{8} \leq d \leq \frac{8m-2}{3}$
6. $n = 8, \frac{48m}{17} \leq d \leq \frac{17m-2}{6}$

(ii) a homogeneous, non-special linear system $\mathcal{L}_d(n^m)$ is (-1)-reducible if and only if one of the following cases occurs:

1. $n = 2, d = 2m - 1$

2. $n = 3, d = 2m - 1$

3. $n = 5, d = \frac{5m-1}{2}$

4. $n = 6, d = \frac{5m-1}{2}$

5. $n = 7, d = \frac{8m-1}{3}$

6. $n = 8, d = \frac{17m-1}{6}$

(iii) *a homogeneous, non-special linear system* $\mathcal{L}_d(n^m)$ *is reducible but not* (-1)*-reducible if and only if one of the following cases occurs:*

1. $n = 1, d = m$

2. $n = 4, d = 2m$

3. $n = 9, d = 3m.$

In particular, if $n \geq 9$ *any homogeneous linear system* $\mathcal{L}_d(n^m)$ *is non-special and irreducible.*

Proof: Part (i) follows from Theorem (2.4) of [CM2] and from Theorem (4.8). Part (ii) follows by the same argument of Theorem (2.4) of [CM2].

In case (iii) we know that either:
(a) $\mathcal{L}_d(n^m)$ consists of a unique elliptic curve E of self-intersection 0, counted with multiplicity $\mu \geq 2$ or;
(b) it is composed of a pencil $\mathcal{P}$ of rational curves of self-intersection 0.

In case (a), by obvious monodromy arguments, E has to contain all the n base points of $\mathcal{L}_d(n^m)$. This forces $n = 9$ and $\mu = m$, thus $d = 3m$.

In case (b), one similarly sees that $\mathcal{P}$ can either be the pencil of lines through a point or the pencil of conics through four points. This finishes the proof of the theorem. $\diamondsuit$

7 Nagata's Conjecture

In this section we prove that Segre's Conjecture implies the following conjecture due to Nagata.

Theorem 7.1 *Suppose that Segre's Conjecture is true. Let* $p_1, \ldots, p_n$ *be general points in the plane. Let* C *be a plane curve of degree* d *having multiplicity at least* m_i *at the point* p_i*. Then for all* $n \geq 10$

$$\sum_{i=1}^{n} m_i < d\sqrt{n}.$$

The theorem (without the Segre Conjecture hypothesis) was proved by M. Nagata for all perfect squares $n = p^2$, $p \geq 4$ [Na]. The technique used is specializations of the points in the plane. As a consequence of this result Nagata produced a counterexample to Hilbert's 14th problem. More recently Z. Ran has given an affirmative answer for a broad selection of integers $n \geq 10$ [Ra]. The proof relies on degenerating the plane to certain reducible surfaces called mosaics. It is also worth mentioning that in [Xu1], G. Xu proves a similar bound, namely that

$$d \geq \frac{\sqrt{n-1}}{n} \sum_{i=1}^{n} m_i.$$

Recently B. Harbourne has made progress on Nagata's Conjecture, see [Ha2].

See also [Xu1] and [MP] for applications to symplectic packings of $B^4(1)$ (the symplectic ball of radius 1), and [Xu2] for applications to multiple-point Seshadri constants.

Proof: If $\sum_{i=1}^{n} m_i \geq d\sqrt{n}$, then there exists an irreducible and reduced component C' of C of degree d' with multiplicity at least m'_i at the point p_i such that

$$\sum_{i=1}^{n} m'_i \geq d'\sqrt{n}.$$

We may therefore assume that the curve C itself is reduced and irreducible. By 4.1 the linear system $\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})$ is non empty and non-special. In particular

$$\dim(\mathcal{L}_d(p_1^{m_1}, ..., p_n^{m_n})) = \frac{d(d+3) - \sum_{i=1}^{n} m_i(m_i+1)}{2} \geq 0.$$

If $\sum_{i=1}^{n} m_i \geq d\sqrt{n}$, then

$$\begin{aligned} 0 &\leq d(d+3) - \sum_{i=1}^{n} m_i(m_i+1) \\ &\leq \frac{(\sum_{i=1}^{n} m_i)^2}{n} + \frac{3\sum_{i=1}^{n} m_i}{\sqrt{n}} - \sum_{i=1}^{n} m_i^2 - \sum_{i=1}^{n} m_i \end{aligned}$$

By Cauchy-Schwartz $(\sum_{i=1}^{n} m_i)^2 \leq n(\sum_{i=1}^{n} m_i^2)$. It follows that

$$0 \leq \sum_{i=1}^{n} m_i \left(\frac{3}{\sqrt{n}} - 1 \right).$$

This is clearly impossible for all $n \geq 10$. ♢

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