SOME REMARKS ON THE OBSTRUCTEDNESS OF CONES OVER CURVES OF LOW GENUS

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1. INTRODUCTION

Let C be a smooth irreducible curve of genus g and L a very ample line bundle on C of degree d. Embedding C in $\mathbb{P}^r = \mathbb{P}H^0(L)^*$ we can consider the cone $X_{C,L} \subset \mathbb{P}^{r+1}$ over C with vertex a point in $\mathbb{P}^{r+1} - \mathbb{P}^r$. When $g = 0$ it was shown by Pinkham ([P]) that rational normal cones are obstructed, that is they represent singular points of their Hilbert schemes, as soon as $r \geq 4$. One of the purposes of the present article is to employ the technique of Gaussian maps to study the obstructedness of cones over curves of *positive* genus g and degree $d >> g$.

For $k \geq 1$ set $R(\omega_C \otimes L^{k-1}, L) = Ker\{H^0(\omega_C \otimes L^{k-1}) \otimes H^0(L) \to H^0(\omega_C \otimes L^k)\}$ and let

$$\Phi_{\omega_C \otimes L^{k-1}, L} : R(\omega_C \otimes L^{k-1}, L) \to H^0(\omega_C^2 \otimes L^k)$$

be the Gaussian map defined locally by $\Phi_{\omega_C \otimes L^{k-1}, L}(s \otimes t) = sdt - tds$. Moreover let $\gamma^k_{C,L} = corank\ \Phi_{\omega_C \otimes L^{k-1}, L}$, and $\gamma_{C,L} = corank\ \Phi_{\omega_C, L}$.

As is well-known now the integers $\gamma^k_{C,L}$ may be used to compute the dimension of the tangent space to the Hilbert scheme at points representing cones $X_{C,L}$ over C (see for example [W1], [St], [T], [CM], [CLM1], [CLM2]).

We will see in section 2 that there is an interesting difference for the obstructedness of such cones between the nonhyperelliptic case for $g \geq 3$ and the hyperelliptic case $g = 1, 2$.

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For $g \geq 3$ and $d >> g$ if C is not hyperelliptic and L is any line bundle on C of degree d, we have $\gamma_{C,L} = 0$ ([W2], [BEL]), and the cone $X_{C,L}$ is unobstructed. On the other hand if C is hyperelliptic and $d >> g$ then $\gamma_{C,L} > 0$ ([St]); in particular if $g = 1, 2$ and $d \geq 3g + 7$, we will show in section 4, that the cone $X_{C,L}$ is obstructed and cannot be smoothed. By taking general hypersurface sections of such cones and using the above fact, we will construct infinitely many examples of *nonreduced* components of the Hilbert scheme of curves in $\mathbb{P}^{r+1}, r \geq 9$ (see Theorem (4.11)). Note that so far all the known examples of nonreduced components of the Hilbert scheme are for curves in $\mathbb{P}^3$ ([M], [GP], [K], [E], [Fl]).

Another interesting feature of the coranks of Gaussian maps is that they give the cohomology $h^0(N_C(-k)), k \geq 1$, of the normal bundle of C in $\mathbb{P}^r$. This in turn, by a theorem of Zak [Z], governs in many cases the existence of higher dimensional varieties having C as their curve section (see for example [CLM1], [CLM2] for Fano varieties, [W1] for K3 surfaces). In section 3 we will further exploit the power of the above technique, by applying it to varieties of degree five having elliptic curve sections. In particular we will extend to the singular case the well-known smooth classification ([Sc], [I], [Fu]). We will then consider surfaces in $\mathbb{P}^r, r \geq 6$ with curve sections of genus at most three. In these cases a straightforward application of the method of Gaussian maps and Zak's theorem does not work. We will instead recover the known classification by means of a projective technique (the "tetragonal lemma" (3.2)).

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2. GLOBAL SECTIONS OF THE NORMAL BUNDLE AND CORANK OF GAUSSIAN MAPS

As above we let C be a smooth irreducible curve of genus $g \geq 0$, L be a line bundle on

C of degree $d \geq 1$ and $\gamma_{C,L} = corank\ \Phi_{\omega_C,L}$. We will first collect some facts about $\gamma_{C,L}$.

Proposition (2.1). *We have the following values for $\gamma_{C,L}$:*

(2.2) If $g = 0$ then $\gamma_{C,L} = \begin{cases} 0 & \text{for } d \leq 3 \\ d-3 & \text{for } d \geq 4 \end{cases}$;

(2.3) If $g = 1$ then $\gamma_{C,L} = d$;

(2.4) If $g = 2$ then $\gamma_{C,L} = 6$ for $d \geq 5$;

(2.5) If $g = 3$ and C is not hyperelliptic we have

$$\gamma_{C,L} = \begin{cases} 0 & \text{for } d \geq 17 \\ 0 & \text{for } L \text{ general and } d \geq 14 \\ \geq 14-d & \text{for } 6 \leq d \leq 13 \end{cases};$$

(2.6) If $g = 4$ and C is not hyperelliptic we have

$$\gamma_{C,L} = \begin{cases} 0 & \text{for } d \geq 19 \\ 0 & \text{for } L \text{ general and } d \geq 15 \\ \geq 15-d & \text{for } 8 \leq d \leq 14 \end{cases};$$

(2.7) If $g = 5$ and C is not trigonal we have

$$\gamma_{C,L} = \begin{cases} 0 & \text{for } d \geq 17 \\ 0 & \text{for } L \text{ general and } d \geq 12 \\ \geq 36-3d & \text{for } 10 \leq d \leq 11 \end{cases};$$

(2.8) If $g \geq 6$ and both C and L are general we have

$$\gamma_{C,L} = 0 \text{ if } d \geq \begin{cases} g+12 & \text{for } 6 \leq g \leq 8 \\ g+9 & \text{for } g \geq 9 \end{cases}.$$

Proof. If $g = 0, 1$ we have $R(\omega_C, L) = 0$, hence $\gamma_{C,L} = h^0(\omega_C^2 \otimes L)$ and this gives (2.2) and (2.3). If $g = 2$, by the base point free pencil trick, we have $R(\omega_C, L) = H^0(\omega_C^{-1} \otimes L)$ hence $\gamma_{C,L} = h^0(\omega_C^2 \otimes L) - h^0(\omega_C^{-1} \otimes L) + dimKer\ \Phi_{\omega_C,L} = d + 3 - (d-3) + dimKer\ \Phi_{\omega_C,L} = 6 + dimKer\ \Phi_{\omega_C,L}$. A straightforward computation in the diagram

$$\begin{tikzcd} H^0(\omega_C^{-1} \otimes L) \arrow[rr, "\Phi"] \arrow[dr, "\cong"'] & & H^0(\omega_C^2 \otimes L) \\ & R(\omega_C, L) \arrow[ur, "\Phi_{\omega_C,L}"'] & \end{tikzcd}$$

gives that, if $H^0(\omega_C) =< s, t >$, we have $\Phi(\sigma) = 2\sigma\Phi_{\omega_C,\omega_C}(s \otimes t)$, and $\Phi_{\omega_C,\omega_C}(s \otimes t) \neq 0$, hence both Φ and $\Phi_{\omega_C,L}$ are injective and (2.4) follows. To see (2.5), (2.6) and (2.7) consider the canonical embedding $C \subset \mathbb{P}^{g-1}$. As is well known (see [W2]), from the exact sequence

$$0 \to N^*_{C/\mathbb{P}^{g-1}} \otimes \omega_C \otimes L \to \Omega^1_{\mathbb{P}^{g-1}|C} \otimes \omega_C \otimes L \to \omega_C^2 \otimes L \to 0$$

we get

(2.9) $\quad H^0(\Omega^1_{\mathbb{P}^{g-1}|C} \otimes \omega_C \otimes L) \xrightarrow{\Phi_{\omega_C,L}} H^0(\omega_C^2 \otimes L) \to H^1(N^*_{C/\mathbb{P}^{g-1}} \otimes \omega_C \otimes L) \to$

$\to H^1(\Omega^1_{\mathbb{P}^{g-1}|C} \otimes \omega_C \otimes L)$

and in the cases at hand we will show that

(2.10) $H^1(\Omega^1_{\mathbb{P}^{g-1}|C} \otimes \omega_C \otimes L) = 0$ and therefore

(2.11) $\gamma_{C,L} = h^1(N^*_{C/\mathbb{P}^{g-1}} \otimes \omega_C \otimes L)$.

To see (2.10) consider the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^{g-1}|C} \otimes \omega_C \otimes L \to H^0(\omega_C) \otimes L \to \omega_C \otimes L \to 0$$

and notice that, with the given hypotheses on d, we have $H^1(L) = 0$ and the multiplication map $H^0(\omega_C) \otimes H^0(L) \to H^0(\omega_C \otimes L)$ is surjective (by [C2]), hence (2.10).

When $g = 3$ by (2.11) we see that $\gamma_{C,L} = h^1(L \otimes \omega_C^{-3}) = h^0(L^{-1} \otimes \omega_C^4)$ and this is zero for $d \geq 17$ or is at least $14 - d$ for $6 \leq d \leq 13$. For $14 \leq d \leq 16$ and L general in $Pic^d C$ we have that $\gamma_{C,L} = 0$ since $dim\ Pic^d C = 3$ while $dim\{L : h^0(L^{-1} \otimes \omega_C^4) \neq 0\} = 16 - d$. This gives (2.5). When $g = 4$ we get by (2.11) that $\gamma_{C,L} = h^1(L \otimes (\omega_C^{-1} \oplus \omega_C^{-2})) = h^0(L^{-1} \otimes \omega_C^2) + h^0(L^{-1} \otimes \omega_C^3)$ and this is zero for $d \geq 19$ or is at least $15 - d$ for $8 \leq d \leq 14$. For $15 \leq d \leq 18$ and L general in $Pic^d C$ we have that $\gamma_{C,L} = 0$ since $dim Pic^d C = 4$ while $dim\{L : h^0(L^{-1} \otimes \omega_C^3) \neq 0\} = 18 - d$. This proves (2.6). Similarly when $g = 5$, again by (2.11), we have that $\gamma_{C,L} = h^1(L \otimes (\omega_C^{-1})^{\oplus 3}) = 3h^0(L^{-1} \otimes \omega_C^2)$ and this is zero for $d \geq 17$ or is at least $36 - 3d$ for $10 \leq d \leq 11$. For $12 \leq d \leq 16$ and L general in $Pic^d C$ we have that $\gamma_{C,L} = 0$ since $dim Pic^d C = 5$ while $dim\{L : h^0(L^{-1} \otimes \omega_C^2) \neq 0\} = 16 - d$. Hence (2.7) follows. Finally (2.8) was proved in [L, Corollary (1.7)]. ■

We turn now to the relation between the corank of Gaussian maps and the obstructedness of cones. With notation as above, suppose from now on that L is very ample, $C \subset \mathbb{P}^r = \mathbb{P}H^0(L)^*$ and let $X_{C,L}$ be a cone over C in $\mathbb{P}^{r+1}$ with vertex a point $P \in \mathbb{P}^{r+1} - \mathbb{P}^r$.

We recall for the reader's convenience the connection between cohomology of the normal bundle and corank of Gaussian maps. Let $N_C, N_{X_{C,L}}$ be the normal bundles of $C \subset \mathbb{P}^r$ and of $X_{C,L} \subset \mathbb{P}^{r+1}$, respectively. Then

Proposition (2.12). *Suppose $g \geq 1$. Then:*

(2.13) $h^0(N_C(-1)) = r + 1 + \gamma_{C,L}$;

(2.14) $h^0(N_C(-k)) = dimCoker\ \Phi_{\omega_C \otimes L^{k-1},L}$ *for every* $k \geq 2$;

(2.15) If $\gamma_{C,L} = 0$ *then* $h^0(N_C(-k)) = 0$ *for every* $k \geq 2$;

(2.16) $h^0(N_{X_{C,L}}) \leq \sum_{k \geq 0} h^0(N_C(-k))$;

(2.17) If $C \subset I\!P^r = I\!PH^0(L)^*$ *is projectively normal we have equality in (2.16).*

Proof. (2.13) and (2.14) are in [CM, Proposition 1.2]. Of course (2.15) follows for $k \geq 3$ as soon as we prove it for $k = 2$. To this end from the diagram

$$\begin{array}{ccc} R(\omega_C, L) \otimes H^0(L) & \xrightarrow{\Phi_{\omega_C,L} \otimes id_{H^0(L)}} & H^0(\omega_C^2 \otimes L) \otimes H^0(L) \\ \downarrow & & \downarrow \mu \\ R(\omega_C \otimes L, L) & \xrightarrow{\Phi_{\omega_C \otimes L,L}} & H^0(\omega_C^2 \otimes L^2) \end{array}$$

we deduce that $\Phi_{\omega_C \otimes L,L}$ is surjective since $\Phi_{\omega_C,L}$ is by hypothesis and so is μ (see for example [G, Theorem (4.e.1)]). Hence $H^0(N_C(-2)) = 0$ by (2.14).

To see (2.16) set $X = X_{C,L}$ and notice that $N_{X|C} \cong N_C$, and therefore the exact sequence

$$0 \to N_X(-h-1) \to N_X(-h) \to N_C(-h) \to 0$$

implies that $h^0(N_X(-h)) \leq h^0(N_X(-h-1)) + h^0(N_C(-h))$, and applying this successively we get (2.16) since $h^0(N_X(-j)) = 0$ for $j >> 0$. Finally (2.17) follows by standard facts (see for example [CM, (4.2)]). ■

Now let W be a component of the Hilbert scheme $H_{d,g,r}$ of curves of degree d and genus g in $I\!P^r$ such that $W \ni [C]$, the point representing C, and denote by $\mathcal{H}(W)$ the family of cones in $I\!P^{r+1}$ over curves representing points of W. We have the following general fact

Proposition (2.18). *Suppose that* $g \geq 3$ *and a general point* $[C_\eta]$ *of* W *is a smooth point such that* $\gamma_{C_\eta,\mathcal{O}_{C_\eta}(1)} = 0$. *Then*

(a) $\mathcal{H}(W)$ *is a generically smooth component of the Hilbert scheme of surfaces of degree* d *in* $I\!P^{r+1}$ *and* $dim\mathcal{H}(W) = h^0(N_{C_\eta}) + r + 1$. *In particular* $dim\mathcal{H}(W) = (r+1)(d+1) + (r-3)(1-g)$ *if* $H^1(N_{C_\eta}) = 0$;

(b) If $\gamma_{C,L} = 0$ *and* $[C]$ *is a smooth point of* W, *then* $X_{C,L}$ *is unobstructed;*

(c) If $C \subset I\!P^r = I\!PH^0(L)^*$ *is projectively normal and* $X_{C,L}$ *is unobstructed, then* $\gamma_{C,L} = 0$.

Proof. First suppose that $\gamma_{C,L} = 0$ and that $[C]$ is a smooth point of W. Let V be a component of the Hilbert scheme of surfaces of degree d in $\mathbb{P}^{r+1}$ such that $V \supseteq \mathcal{H}(W)$. We have

$$h^0(N_C) + r + 1 = dim W + r + 1 = dim\mathcal{H}(W) \leq dim V \leq h^0(N_{X_{C,L}}) \leq h^0(N_C) + r + 1$$

where the last inequality follows by (2.16), (2.15) and (2.13). Hence $V = \mathcal{H}(W)$ and $X_{C,L}$ is unobstructed. This shows (b). Applying this to C_η we get (a) since $h^0(N_{C_\eta}) + r + 1 = (r+1)(d+1) + (r-3)(1-g)$ when $H^1(N_{C_\eta}) = 0$. Now suppose $C \subset \mathbb{P}^r = \mathbb{P}H^0(L)^*$ is projectively normal and $X_{C,L}$ is unobstructed. Since $\mathcal{H}(W)$ is a component of the Hilbert scheme, it must be the only one containing $[X_{C,L}]$. If $\gamma_{C,L} > 0$ we would have

$$\begin{aligned} dim\mathcal{H}(W) &= h^0(N_{C_\eta}) + r + 1 \leq h^0(N_C) + r + 1 < \\ &< h^0(N_C) + r + 1 + \gamma_{C,L} + \sum_{k \geq 2} h^0(N_C(-k)) = h^0(N_{X_{C,L}}) \end{aligned}$$

by (2.17). But this shows that $X_{C,L}$ is obstructed. Hence (c) is proved. ∎

(2.19) *Remark.* Even for line bundles of high degree it is possible that $\gamma_{C,L} > 0$ for a given pair (C, L) while $\gamma_{C_\eta, L_\eta} = 0$ for the general pair (C_η, L_η). For example on a smooth nonhyperelliptic curve C of genus $g \geq 9$ take P a general point of C and $L = \omega_C(2P)$. It is shown in [L] that $\gamma_{C,L} > 0$; on the other hand by Proposition (2.1) we have $\gamma_{C_\eta, L_\eta} = 0$ since $deg L_\eta = 2g \geq g + 9$. Other examples can be deduced from [St], [T].

Pairing Propositions (2.1) and (2.18) we get

Corollary (2.20). *Let C be a smooth irreducible curve of genus $g \geq 3$, L a line bundle on C of degree d such that either*

(a) $d \geq 16g - 2g^2 - 13$ for $3 \leq g \leq 5$ or

(b) $d \geq 15g - 2g^2 - 13$ for $3 \leq g \leq 5$ and L is general, C is not hyperelliptic and also not trigonal if $g = 5$, or

(c) $d \geq \begin{cases} g + 12 & \text{for } 6 \leq g \leq 8 \\ g + 9 & \text{for } g \geq 9 \end{cases}$ and both C and L are general.

Then $X_{C,L}$ is unobstructed and $\mathcal{H}(W)$ is a generically smooth component of the Hilbert scheme of dimension $(r+1)(d+1) + (r-3)(1-g)$ where $r = d - g$.

Proof. By Proposition (2.1) under hypothesis (a), (b) or (c), we have $\gamma_{C,L} = 0$. Moreover in all cases $H^1(L) = 0$, hence we can apply (a) and (b) of Proposition (2.18). ∎

3. REMARKS ON THE CLASSIFICATION OF VARIETIES WITH ELLIPTIC CURVE SECTIONS

Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety of dimension $n \geq 2$ with $dimSingX^n \leq n-2$, degree 5, whose curve sections are smooth elliptic normal curves. If H is the hyperplane divisor of X^n we have $-K_{X^n} = (n-1)H$ and $N = n+3$.

Well-known examples of such varieties are the linear sections of the Grassmann variety $\mathbb{G}(1,4) \subset \mathbb{P}^9$ in its Plücker embedding. Just as in [CLM2], we will show that the classification of the varieties X^n as above can be recovered by means of deformations to cones and the theorem of Zak, whose statement we recall here. A smooth nondegenerate variety $Y \subset \mathbb{P}^m$ is said to be *k-extendable* if there exists a variety $Z \subset \mathbb{P}^{m+k}$ that is not a cone and such that $Y = Z \cap \mathbb{P}^m$. Zak's theorem ([Z]) says that if *codim* $Y \geq 2$ and $h^0(N_Y(-1)) = m+1$ then Y is not 1-extendable; also if $k \geq 2, h^0(N_Y(-1)) \leq m+k$ and $h^0(N_Y(-2)) = 0$, then Y is not k-extendable.

In the Hilbert scheme of nondegenerate varieties $X^n \subset \mathbb{P}^{n+3}$ of dimension $n \geq 2$ with $dimSingX^n \leq n-2$, degree 5 and such that $-K_{X^n} = (n-1)H$, we let $\mathcal{X}_n$ be the open subset parametrizing varieties X^n that are not cones over an elliptic curve. Then

Proposition (3.1).

(a) $\mathcal{X}_n \neq \emptyset$ *if and only if* $n \leq 6$;

(b) For $n \leq 6, \mathcal{X}_n$ *is irreducible and the family of linear sections of* $\mathbb{G}(1,4) \subset \mathbb{P}^9$ *forms a dense open subset of smooth points of* $\mathcal{X}_n$.

Proof. By (2.3) and (2.13) if $E \subset \mathbb{P}^4$ is a smooth curve section of X^n, we have that $h^0(N_{E/\mathbb{P}^4}(-1)) = 10$. Also it is easily seen that $h^0(N_{E/\mathbb{P}^4}(-2)) = 0$, for example using (2.14) and the surjectivity of $\Phi_{L,L}$ (see [BEL]). Hence Zak's theorem (see [Z]) implies that $n \leq 6$. On the other hand $dim\mathbb{G}(1,4) = 6$ and $-K_{\mathbb{G}(1,4)} = 5H$, where H is the Plücker divisor, hence $\mathcal{X}_6 \neq \emptyset$ and so is $\mathcal{X}_n, n \leq 5$, since a hyperplane section of an X^n is a X^{n-1}. This proves (a). To see (b) we will use an argument similar to the one in [CLM2], in the proof of theorems (3.2) and (3.11). Since the varieties $X^n, n \geq 2$, are projectively Cohen-Macaulay (because E is), they flatly degenerate to the n-dimensional cone $\widehat{E}$ over their general curve section E. Of course the locus of such cones is irreducible, hence (b) follows if we show that these cones are smooth points of the closure of $\mathcal{X}_n$. To this end we will prove, as in [CLM2], that $h^0(N_{\widehat{E}})$ is bounded above by the dimension of the family of the known examples, that is linear sections of $\mathbb{G}(1,4) \subset I\!P^9$. Since $h^0(N_E(-k)) = 0$ for every $k \geq 2$, we have

$$h^0(N_{\widehat{E}}) = h^0(N_E) + (n-1)h^0(N_E(-1)) = 25 + 10(n-1).$$

On the other hand the family of $X^n \subset I\!P^{n+3}$ obtained by linear sections of $\mathbb{G}(1,4) \subset I\!P^9$ has dimension

$dim\mathbb{G}(n+3,9) + dim Aut I\!P^{n+3} - dim \mathrm{Aut}\mathbb{G}(1,4) = (n+4)(6-n) + (n+4)^2 - 25 = 15 + 10n.$

■

Note that the case $n = 2$ of Proposition (3.1) is the one of Del Pezzo surfaces of degree 5 in $I\!P^5$. Suppose now $S \subset I\!P^r, r \geq 6$ is a (not necessarily smooth) surface whose general hyperplane section is a smooth elliptic normal curve E. We have $degE = r$, hence S is a surface of degree r in $I\!P^r$ nonsingular in codimension one. In this case a straightforward application of the methods of [CLM2] does not allow to recover the family of Del Pezzo surfaces. In fact by (2.3) and (2.13) we have $h^0(N_E(-1)) = 2r$, hence Zak's theorem does not rule out the existence of such surfaces $S \subset I\!P^r$ for $r \geq 10$. Moreover even for $6 \leq r \leq 9$ the upper bound on $h^0(N_{\widehat{E}})$ is larger than the dimension of the family of Del Pezzo surfaces. Also note that this implies that the cones over elliptic normal curves are obstructed for $6 \leq r \leq 9$; in section 4 we will see that they are also obstructed for $r \geq 10$.

We will then use another strategy to recover the classification. Recall that E is projectively Cohen-Macaulay, its homogeneous ideal is generated by quadrics, the relations among them being minimally generated by linear ones. The same then holds for S. Let $C = S \cap Q$ be a smooth quadric section of S. We have that C is a canonical curve whose homogeneous ideal is generated by quadrics and therefore by Petri's theorem C is not trigonal nor isomorphic to a plane quintic. Hence $Cliff(C) \geq 2$, where $Cliff(C)$ is the Clifford index of C. Moreover the Koszul relations among Q and the quadrics generating the ideal of S give rise to nonlinear syzygies among the generators of the ideal of C, relations that do not depend on the linear ones (or in other words $h^0(N_C(-2)) \neq 0$). By a result of Schreyer [S1] and Voisin [V], we must have $Cliff(C) = 2$, that is C is either tetragonal or isomorphic to a plane sextic.

This fact will allow us to classify S. The key point is given by the following

Lemma (3.2) (tetragonal lemma). *Let $S \subset \mathbb{P}^r, r \geq 5$ be a surface, nonsingular in codimension one, not a cone, whose general hyperplane section has genus g, degree $d \geq 2g+3$ and is linearly normal. If there is a linear system of curves on S, of dimension at least 1, whose general element D is smooth, irreducible, special and tetragonal, then S is either a rational normal scroll (i.e. $g=0$) or it lies on a rational normal threefold scroll of planes, each meeting S in a conic.*

Proof. First notice that S is cut out by quadrics because its general hyperplane section is projectively normal and is cut out by quadrics, since $d \geq 2g+3$ [G]. Take $p_1+\ldots+p_4$ a general divisor of the g^1_4 on D. The span of this divisor in $\mathbb{P}^r$ is at most a 2-plane π. If it is a line, then S is a rational normal scroll. Suppose now π is a 2-plane. Consider the hyperplane sections $\{H_t\}$ of S through π. Suppose that a general H_t is irreducible. By monodromy the behaviour of H_t at the points $p_1,\ldots,p_4$ is the same, and we claim that $p_1,\ldots,p_4$ are smooth points of H_t. Otherwise H_t would be tangent at $p_1,\ldots,p_4$ and therefore π would be tangent to S at these four points. Now let D and p_1 vary. Since p_1 describes the whole surface, we conclude that the general tangent plane to the surface is tangent at four points, a contradiction, since the Gauss map is birational. By the hypothesis $d \geq 2g+3$, if H_t is irreducible, we have that $H^1(\mathcal{O}_{H_t}(1)(-p_1-\ldots-p_4))=0$ and hence $dim|\mathcal{O}_{H_t}(1)(-p_1-\ldots-p_4)| = d-4-g$ while the $H_{t'}$ cut out, away from $p_1,\ldots,p_4$, a linear series on H_t of dimension at least $r-3=d-g-3$. This contradiction shows that a general H_t must be reducible. Therefore, by Bertini's theorem, there is a fixed component F of the linear system $\{H_t\}$, and $F \subset \pi$ unless $\{H_t\}$ is composed with a pencil. In the latter case, since the dimension of $\{H_t\}$ is at least $r-3 \geq 2$, we have that H_t has to contain the tangent plane to S at each point $p_i, i=1,\ldots,4$, for all t's and this, as we saw, is impossible. Since S is cut out by quadrics, then the degree of F is at most 2 and F has to contain $p_1,\ldots,p_4$. If F is a line we conclude that S is a rational normal scroll. If F is a conic, and is reducible, then S is a scroll. Arguing as we did above, we see that the lines of the scroll are pairwise coplanar, hence S is a cone, which we excluded. If F is an irreducible conic, then S is contained in the rational normal threefold scroll of planes $\bigcup_{p_1+\ldots+p_4\in g^1_4} < p_1+\ldots+p_4 >$. ■

We will use the tetragonal lemma to recover the classification of Del Pezzo surfaces.

Proposition (3.3). *Let $S \subset \mathbb{P}^r, r \geq 6$ be a surface whose general hyperplane section is a smooth elliptic normal curve E. Then S is either an elliptic normal cone or $r \leq 8$ and S is a divisor of class $2H+(4-r)R$ on a rational normal threefold scroll of planes $X \subset \mathbb{P}^r$, with hyperplane section H and ruling R, or S is the Veronese embedding $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ given by the cubics.*

Proof. Notice that if S is a cone then it is an elliptic normal cone. From now on we suppose that S is not a cone. We claim that either S lies on a rational normal threefold scroll of planes, each meeting S in a conic, or $r=9$ and S is smooth. In fact if S has a singular point P, since the ideal of S is generated by quadrics, the projection $S' \subset \mathbb{P}^{r-1}$ of S from P is a surface of degree at most $r-2$, hence $degS' = r-2$ and S' contains a one dimensional family of lines $\{L_t\}$. Set $\pi_t =< P, L_t >$ and $F_t = \pi_t \cap S$. Then $degF_t \leq 2$ since the ideal of S is generated by quadrics and S lies on a rational normal threefold scroll of planes, each meeting S in a conic. If S is nonsingular, by the discussion at the beginning of this section, we know that a smooth quadric section C of S is either tetragonal or isomorphic to a plane sextic. In the former case we can apply the tetragonal lemma since $H^1(\mathcal{O}_C(1)) = H^1(\omega_C) \neq 0$, concluding again that S lies on a rational normal threefold scroll of planes, each meeting S in a conic. When S lies on a scroll as above, pulling back to the normalization, we have $S \sim aH+bR$ and $2 = degR\cap S = R\cdot S\cdot H = R\cdot(aH^2+bH\cdot R) = a$ since $R^2=0, H^2\cdot R=1$. Moreover $r = degS = S\cdot H^2 = 2(r-2)+b$, hence $b = 4-r$. By [S2, (6.3)] we have $2degX + 3b \geq 0$, that is $r \leq 8$. It remains the case where smooth quadric sections C of S are isomorphic to a plane sextic. Then $r=9$ and by what we just showed, S must be smooth. Moreover $K_S = -H, P_2(S) = h^0(\mathcal{O}_S(-2)) = 0$ and $q(S) = h^1(\mathcal{O}_S(-1)) = 0$, therefore S is rational. Also S is minimal, since by Noether's formula we get $b_2(S) = h^{1,1}(S) = 1$, hence $S = v_3(\mathbb{P}^2)$. ■

(3.4) *Remarks.*

(i) With the tetragonal lemma it is easily recovered also the classification of surfaces with sectional genus 2. Let $S \subset \mathbb{P}^r, r \geq 6$ be a surface whose general hyperplane section is a smooth linearly normal curve C_1 of genus 2. Then S is either a cone over C_1, or $r \leq 11$

and S is a divisor of class $2H + (5 - r)R$ on a rational normal threefold scroll of planes $X \subset I\!P^r$, with hyperplane section H and ruling R.

Proof. Let $C = S \cap Q$ be a smooth quadric section of S. We have $degC = 2r+2, g(C) = r+4$ and C is linearly normal and special since $h^1(\mathcal{O}_C(1)) = 2$. Moreover $deg\omega_C(-1) = 4$ and therefore S has a linear system of dimension at least 2, whose general element is smooth, irreducible, special and tetragonal (of course C is not trigonal, since its ideal is generated by quadrics). Since $degS = r + 1 \geq 7$, by Lemma (3.2) we have that, if S is not a cone, then $S \sim aH + bR$ on a rational normal threefold scroll of planes $X \subset I\!P^r$ of degree $r - 2$, and each plane meets S in a conic. Therefore, as in the proof of Proposition (3.3), we get $a = 2$ and $b = 5 - r$. By [S2, (6.3)] we have $2degX + 3b \geq 0$, that is $r \leq 11$. ∎

(ii) Now let $S \subset I\!P^r, r \geq 6$ be a surface whose general hyperplane section is a smooth linearly normal curve C_1 of genus 3. Then S is either a cone over C_1, or $r \leq 14$ and S is the Veronese embedding $v_4(I\!P^2) \subset I\!P^{14}$ given by the quartics or a projection of it in $I\!P^{14-r}$ or S is a quadric section of the cone in $I\!P^6$ over the Veronese surface $v_2(I\!P^2) \subset I\!P^5$. The proof of this is similar to the one above and will be omitted.

(iii) Suppose now $S \subset I\!P^r, r \geq 4$, is a surface of degree d whose general hyperplane section is a smooth linearly normal curve C of genus $g \geq 3$. As already observed by many authors ([BEL], [BC]), by using Zak's theorem ([Z]), one can obtain an upper bound for d. If C is hyperelliptic or trigonal this is well known (see [Se1], [Se2], [Fa]). If $Cliff(C) = 2$ then $d \leq 4g - 4$, while if $Cliff(C) \geq 3$ then $d \leq 4g - 3Cliff(C)$. The above bounds follow by [BEL], since, if d is larger, then $\gamma_{C,\mathcal{O}_C(1)} = 0$.

4. NONREDUCED COMPONENTS OF THE HILBERT SCHEME OF CURVES

Let C be a smooth irreducible curve of genus $g \geq 1$ and L a line bundle on C of degree d such that $d \geq 10$ for $g = 1, d \geq 4g + 5$ for $g \geq 2$. As opposite to the non hyperelliptic case $g \geq 3$, we saw in Proposition (2.1) that in case $g = 1, 2$, we have $\gamma_{C,L} > 0$. Even though Proposition (2.18) does not apply, we will see here that in fact the cone $X = X_{C,L}$ over C with vertex a point $P \in I\!P^{r+1} - I\!P^r$ is obstructed if $g = 1, 2$.

Note that L is very ample, nonspecial, and, by well-known results, $C \subset I\!P^r = I\!PH^0(L)^*$ is projectively normal, its ideal is generated by quadrics and the relations among them are linear ([G]). Now consider a general hypersurface F_n in $I\!P^{r+1}$ of degree $n \geq 4$ and let $\Gamma^n = X \cap F_n$. The nonreduced components of the Hilbert scheme will be obtained by the family of such Γ^n.

Let us first collect some information on X and Γ^n.

Proposition (4.1).

(4.2) Γ^n is a projectively normal smooth curve of degree nd and genus $p_n = ng + d\frac{n(n-1)}{2} - n + 1$;

(4.3) Let $\mathcal{F} = \{\Gamma_t\}_{t \in D}$ be a flat family of projective curves parametrized by a smooth irreducible variety D such that:

(i) $\mathcal{F}$ is a projective family of curves in $I\!P^{r+1}$;

(ii) there exists a closed point t_0 in D such that Γ_{t_0} is a smooth curve of degree nd and genus $p_n = ng + d\frac{n(n-1)}{2} - n + 1$, complete intersection of a hypersurface of degree $n \geq 4$ with a cone over a projectively normal, nonspecial curve C of degree d and genus g.

Then there is a neighborhood U of t_0 in D in the Zariski topology such that, for all closed points $t \in U$, Γ_t is again the complete intersection of a hypersurface of degree n with a cone over a projectively normal, nonspecial curve of degree d and genus g.

Proof. Let Δ be a general hyperplane section of Γ^n; we may think of Δ as a divisor on C belonging to the linear system $|\mathcal{O}_C(n)|$. If we denote by $h_\Delta(t)$ the Hilbert function of Δ, we find

$$h_\Delta(t) = \begin{cases} h^0(C, \mathcal{O}_C(t)) = td - g + 1, & \text{for t = 1,..., n-1} \\ h^0(C, \mathcal{O}_C(n)) - 1 = nd - g & \text{for t = n} \\ nd & \text{for } t \geq n+1 \end{cases} .$$

Therefore we have

$$\sum_{t=1}^{n} (nd - h_\Delta(t)) = \sum_{t=1}^{n} [g + (n-t)d - 1] + 1 = ng + d\frac{n(n-1)}{2} - n + 1 = p_n$$

which, in view of [C1], remark (1.8), (ii), implies that Γ^n is projectively normal. To see (4.3) we can assume D to be affine, thus $D = Spec(A)$. We have then a family $Y \subset I\!P_A^{r+1} = Proj\, A[x_0, ..., x_{r+1}]$ over $Spec(A)$. If $\mathcal{I}_Y$ is the ideal sheaf of Y in $I\!P^{r+1}$, then $\mathcal{I}(Y) = \bigoplus_{n \in \mathbb{N}} H^0(\mathcal{I}_Y(n))$ is the homogeneous ideal of Y in $S = A[x_0, ..., x_{r+1}]$. Consider the homogeneous ideal $\mathcal{I} \subset \mathcal{I}(Y)$ generated by $\mathcal{I}(Y)_2 = H^0(\mathcal{I}_Y(2))$, and look at the scheme $W = Proj(S/\mathcal{I})$. There is a commutative diagram

$$\begin{array}{ccl} Y \hookrightarrow W \subset & \mathbb{P}^{r+1}_A = Proj\, A[x_0, ..., x_{r+1}] \\ \searrow \downarrow & \\ Spec\,(A) & \end{array}$$

and restricting to the local ring of D at t_0 we find that the scheme W is flat over the generic point ξ of D (the proof of this is similar to the one of proposition (1.6) of [C1] and will be omitted). Therefore W is flat over a neighborhood U of t_0 in D, thus giving a flat family of surfaces X_t, with central fiber the cone X. Notice that since C is smooth and projectively normal, then X is normal at the vertex P. For a general point ξ of D, X_ξ has to be singular (see [P, theorem (7.5)]), and has normal singularities (see [EGA, theorem 12.2.4]). On the other hand, by theorem 2, chapter 1 of Horowitz's thesis [Ho], X_ξ (which is irreducible and nondegenerate, as well as X) is a scroll. Therefore X_ξ has to be a cone, because these are the only scrolls which may have normal singularities as we will see in the Claim (4.4) below. In any case, since the family $\{\Gamma_t\}$ is very flat, lifting the equation of the hypersurface F_n, we see that also Γ_t is the complete intersection of a hypersurface of degree n with a cone over a projectively normal, nonspecial curve of degree d and genus g.

Claim (4.4). *Let T be an irreducible, nondegenerate normal surface in $\mathbb{P}^r, r \geq 3$ and assume that T is a scroll (that is T is ruled by lines). If T is singular then T is a cone over a smooth curve.*

Proof. Let Q be a singular point of T and let H be the hyperplane section of T with a general hyperplane passing through Q. We shall separately discuss two cases.

Case 1: H is reducible.

Let $\pi : \widetilde{T} \to T$ be the minimal desingularization of T and let E be the divisor contracted by π in Q. Let moreover Σ be the linear system of divisors on $\widetilde{T}$ which are the pull-backs on $\widetilde{T}$ of the hyperplanes of $\mathbb{P}^r$ passing through Q. Clearly the base locus of Σ coincides with E and our hypothesis on H yields that the movable part of Σ is composed with a pencil $\mathcal{P}$. Now let R be a general point of T and $\widetilde{R}$ the corresponding point of $\widetilde{T}$. All curves of Σ passing through $\widetilde{R}$ must contain the curve of $\mathcal{P}$ passing through $\widetilde{R}$. This curve, in turn, has to be the pull-back on $\widetilde{T}$ of some curve on T contained in all the hyperplanes passing through Q and R. Whence clearly the general curve in $\mathcal{P}$ is the pull-back on $\widetilde{T}$

of a line contained in T and passing through Q. This implies that T is a cone with Q as vertex. The directrix of the cone has to be smooth by the normality of T.

Case 2: H is irreducible.

It is easy to construct in this case a flat family $\{H_t\}_{t\in B}$ such that:

(i) B is a smooth curve;

(ii) for every $t \in B, H_t$ is a hyperplane section of T;

(iii) there exists a closed point $t_1 \in B$ such that for all $t \in B - \{t_1\}, H_t$ is a smooth irreducible hyperplane section of T;

(iv) in correspondence with the point $t_1 \in B$ one has that H_{t_1} is a general hyperplane section of T through Q;

(v) H_{t_1} is reduced (because of the S_2 property of T at Q) irreducible and singular only at Q.

In view of our hypothesis on H, the lines of the scroll do not all pass through Q (otherwise T would be a cone and H reducible). But this clearly implies that H_t, for $t \neq t_1$, is birational to H_{t_1}, a contradiction since the geometric genus of H_{t_1} has to be strictly less than the one of H_t (see [Hi]).

This proves Claim (4.4) and hence also concludes the proof of Proposition (4.1). ∎

Now let N_X and N_{Γ^n} be the normal bundles of X and Γ^n, respectively in $\mathbb{P}^{r+1}$; $N_{\Gamma^n/X}$ the normal bundle of Γ^n in X. We have

Proposition (4.5). *Let $C \subset \mathbb{P}^r = \mathbb{P}H^0(L)^*$ be a smooth irreducible curve of genus $g \geq 1, L$ a line bundle on C of degree $d \geq \begin{cases} 10 & \text{for } g=1 \\ 4g+5 & \text{for } g \geq 2 \end{cases}$ and let X be the cone over C with vertex a point $P \in \mathbb{P}^{r+1} - \mathbb{P}^r$. Then*

(4.6) $H^0(X, N_X(-k)) = 0$ for all $k \geq 2$;

(4.7) $H^1(\Gamma^n, N_{\Gamma^n/X}) = 0$;

(4.8) $h^0(\Gamma^n, N_{\Gamma^n}) = h^0(X, N_X) + h^0(\Gamma^n, N_{\Gamma^n/X})$.

Proof. To see (4.6) first notice that by standard facts (for example applying (2.14) and [BEL]) it follows that

(4.9) $H^0(C, N_C(-k)) = 0$ for all $k \geq 2$.

Of course the assertion (4.6) is true for $k >> 0$. So we may argue by descending induction. From the exact sequence

$$0 \to N_X(-k-1) \to N_X(-k) \to N_C(-k) \to 0$$

we see that (4.6) follows by (4.9). Since $N_{\Gamma^n/X} \cong \mathcal{O}_{\Gamma^n}(n)$, we have

$$deg N_{\Gamma^n/X} = n^2 d > n^2 d - n(d-2g+2) = 2gn + dn(n-1) - 2n = 2p_n - 2$$

hence (4.7). To show (4.8) observe that by (4.6) we have $H^0(X, N_X(-n)) = 0$, hence there is an injection $H^0(X, N_X) \xrightarrow{h} H^0(\Gamma^n, N_{X_{|\Gamma^n}})$. Now consider the exact sequence

$$0 \to H^0(\Gamma^n, N_{\Gamma^n/X}) \to H^0(\Gamma^n, N_{\Gamma^n}) \xrightarrow{f} H^0(\Gamma^n, N_{X_{|\Gamma^n}}) \to 0 \tag{4.10}$$

the map f being surjective by (4.7). By the proof of (4.3) we can deduce the existence of a map ϕ which makes the following diagram commutative

$$\begin{array}{ccc} H^0(\Gamma^n, N_{\Gamma^n}) & \xrightarrow{f} & H^0(\Gamma^n, N_{X_{|\Gamma^n}}) \\ \searrow \phi & & \nearrow h \\ & H^0(X, N_X) & \end{array}$$

as one can see by a local computation. This implies that h is also surjective. The assertion then follows by (4.10). ∎

We shall finally prove the announced result on nonreduced components of the Hilbert scheme.

Theorem (4.11). *Let Γ^n be a smooth complete intersection with a hypersurface of degree $n \geq 4$ of a cone $X = X_{C,L}$ over a smooth irreducible curve C of genus $g = 1, 2$, embedded in $\mathbb{P}^r = \mathbb{P}H^0(L)^*$ by a line bundle L of degree $d \geq 3g+7$. Then Γ^n belongs to a unique irreducible component $\mathcal{V}$ of the Hilbert scheme $H_{nd,p_n,r+1}$, $r = d-g$, such that $\mathcal{V}$ is nonreduced and*

$$dim\mathcal{V} = (r+1)(d+1) + (r-3+n)(1-g) + d\frac{n(n+1)}{2}.$$

Proof. Let $\mathcal{V}$ be a component of the Hilbert scheme containing the point $[\Gamma^n]$ and W the component of the Hilbert scheme containing $[C]$. The proof of (4.3) yields the existence of a morphism $\psi : \mathcal{V}_{red} \to \mathcal{H}(W)_{red}$, which is clearly surjective, its (set-theoretical) general fiber being nothing else than the projective space corresponding to the linear system of curves cut out on a cone by the hypersurfaces of degree n in $\mathbb{P}^{r+1}$. Since for any such smooth curve Γ' on a cone X' we have, as we saw in (4.7), that $H^1(\Gamma', N_{\Gamma'/X'}) = 0$, then $h^0(\Gamma', N_{\Gamma'/X'}) = n^2 d - (ng + d\frac{n(n-1)}{2} - n + 1) + 1 = d\frac{n(n+1)}{2} - (g-1)n$. Therefore we find

$$dim\mathcal{V} = dim\mathcal{H}(W) + d\frac{n(n+1)}{2} - (g-1)n = dimW + r + 1 + d\frac{n(n+1)}{2} - (g-1)n =$$
$$= (r+1)(d+1) + (r-3+n)(1-g) + d\frac{n(n+1)}{2}.$$

Thus the part of the statement concerning the uniqueness of $\mathcal{V}$ and its dimension is proved. Now we notice that, by Proposition (2.1) and by (2.13) and (4.8), we have

$$dim\mathcal{V} = h^0(\Gamma^n, N_{\Gamma^n/X}) + dim\mathcal{H}(W) = h^0(\Gamma^n, N_{\Gamma^n/X}) + h^0(C, N_C) + r + 1 <$$
$$< h^0(\Gamma^n, N_{\Gamma^n/X}) + h^0(C, N_C) + r + 1 + \gamma_{C,L} \leq h^0(\Gamma^n, N_{\Gamma^n/X}) + h^0(X, N_X) =$$
$$= h^0(\Gamma^n, N_{\Gamma^n}) = dimT_{[\Gamma^n]}\mathcal{V}$$

where $T_{[\Gamma^n]}\mathcal{V}$ is the Zariski tangent space to $\mathcal{V}$ at $[\Gamma^n]$. ■

(4.12) *Remark.* We limited ourselves to constructing nonreduced components of the Hilbert scheme of curves that are complete intersection of cones with hypersurfaces. Similar results can be proved also for all curves of sufficiently high degree on such cones and for curves which are algebraically equivalent to a hypersurface intersection plus a line.

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