Bivariate Hermite interpolation via computer algebra and algebraic geometry techniques

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Abstract

The bivariate Hermite interpolation problem deals with the question of finding polynomials in two variables of degree at most d with prescribed values and partial derivatives at a finite set of points p_i $(i = 1 \ldots n)$. At each point p_i, the derivatives in question are all partials of order at most $m_i - 1$, for some fixed integer m_i. This question can be expressed as a linear problem in the coefficients of the desired polynomial; the corresponding homogeneous linear problem is to study the vector space of bivariate polynomials of degree at most d which have all partial derivatives up through order $m_i - 1$ vanish at each point p_i; in this case we say that the polynomial has multiplicity m_i at p_i. The determination of the dimension of this vector space is still an open question in general. When the multiplicities are all equal, to m say, this problem has been attacked by a number of authors, (see e.g. [18], [7], [8], [15]) and there are a number of good conjectures ([16], [9]) on the dimension and general elements of these interpolating spaces. In [8] the determination of the dimension has been solved for $m \leq 12$ and all d and n by a degeneration technique and some ad hoc geometric arguments. Here this technique is applied up through $m = 20$; since it fails in some cases, we resort (in these exceptional cases) to an algorithm which is based on the theory of Gröbner bases. In summary we are able to prove that the dimension of the vector space is the expected one for $13 \leq m \leq 20$.

1 Introduction

In this article we work over the field of complex numbers. Fix n distinct general points $p_1, \ldots, p_n$ in the plane and let $m_1, \ldots, m_n$ be nonnegative integers. Also fix a degree d, and consider the vector space of all polynomials in two variables of degree at most d having multiplicity at least m_i at p_i for each i. This is the space of polynomials $P(x, y)$ such that for each i, and for each (a, b) with

$a + b < m_i$,
$$\frac{\partial^{a+b} P}{\partial x^a \partial y^b}(p_i) = 0.$$
Removing the identically zero polynomial and identifying polynomials which are scalar multiples, we denote the projective space of such (nonzero) polynomials by $\mathcal{L} = \mathcal{L}(m_1, \ldots, m_n) = \mathcal{L}_d(-\sum_{i=1}^n m_i p_i)$. We refer to this, in accordance with the language of algebraic geometry, as a *linear system.* Because the zero sets of a polynomial in two variables forms a plane curve, this is a linear system of plane curves. The vanishing of the polynomial P at a point p is exactly the multiplicity one condition. For this reason higher multiplicity conditions expressed as the additional partial derivative vanishings have been referred to in the algebraic geometry literature as "fat point" vanishing. The dimension of the (projective) space of all nonzero polynomials of degree at most d is $d(d+3)/2$. (This is one less than the vector space dimension.) The number of partial derivatives being asked to be zero at a point of multiplicity m is $m(m+1)/2$, and each of these conditions is a linear condition on the coefficients of the polynomial. Therefore we define the *virtual dimension* of $\mathcal{L}$ to be
$$\nu = \nu_d(-\sum_{i=1}^n m_i p_i) = \frac{d(d+3)}{2} - \sum_{i=1}^n \frac{m_i(m_i+1)}{2}.$$
The actual dimension of the linear system cannot be less than -1 and so we define the *expected dimension* to be
$$e = e_d(-\sum_{i=1}^n m_i p_i) = \max\{-1, \nu\}.$$
(A projective space of dimension zero is simply one point; a projective space of dimension -1 is empty.) Here we are interested in the *general dimensionality problem* for points in the plane, i.e. we want to know if the dimension of $\mathcal{L}$ is equal to the expected dimension. If this happens, we say that the linear system $\mathcal{L}$ is *non-special.* Equivalently, we are interested in the classification of all linear systems $\mathcal{L}(m_1, \ldots, m_n)$ which are *special*, that is, whose dimension exceeds the expected dimension. In this article we focus on the case in which all of the multiplicities are equal, to m say. In this case we denote the system by $\mathcal{L} = \mathcal{L}_d(m^n)$ and call it *homogeneous.* Whether $\mathcal{L}$ has the expected dimension or not certainly depends on the position of the points, even if all the multiplicities are one. Indeed, if the points are in some very special position, the dimension of $\mathcal{L}$ can be very large. However it is elementary that for an open dense set in the parameter space of the set of n points, the dimension achieves a minimum value, and it is this dimension that is to be compared with the expected dimension. If the points are such that the dimension is this minimum possible dimension, we say that the points are *in general position* If the points are in general position and the multiplicities are one, then the dimension of the linear system $\mathcal{L}$ is always equal to the expected dimension (for example, see [13, 21]). In [8] Ciliberto and Miranda reformulate a conjecture of Harbourne [14] and Hirschowitz [16] about the dimension of $\mathcal{L}_d(m^n)$ when the points

are in general position and prove this conjecture for all $m \leq 12$. (We present this conjecture below.) They use a degeneration technique that was developed in [7] and that produces an algorithm which has been implemented in the language C [8]. In this range ($m \leq 12$), the degeneration technique failed in some cases, and various ad hoc geometric arguments were used to complete the proof. The computation of the dimension of $\mathcal{L}$ is equivalent to the computation of the Hilbert function of the graded algebra S/I where I is the ideal of n general fat points of multiplicities m. In this article we replace the geometric arguments used in the previous work when the degeneration technique fails by a computer algebra computation of the Hilbert function. For computing the Hilbert function of fat points we rework the Hermite polynomial interpolation problem in such a way that we can apply the theory of Gröbner bases. As a result we can produce an algorithm, based on both, the recursion provided by the degeneration technique and the Gröbner bases computation, which, given an integer $m > 0$, verifies whether all linear systems of the type $\mathcal{L}_d(m^n)$ verify the Harbourne-Hirschowitz conjecture. The related computer program has been implemented, tested, and gave an affirmative answer, for all m between 13 and 20. This way we have been able to prove the Harbourne-Hirschowitz conjecture for $m \leq 20$. It is noteworthy that in [18, Theorem 8] the general dimensionality problem for points in the plane is solved (for $m \leq 4$) using a detailed study of the ranks of the relevant matrices. The computer algebra computation presented below has its origins in 1982, when Buchberger and Möller [23] described an algorithm for computing reduced Gröbner bases of ideals of n points on $\mathbb{A}^r$ in a time that is polynomial in n and in r. For many authors this algorithm has been the starting point for making computations with zero-dimensional varieties in polynomial time (see [1, 2 10, 22, 24, 25, 26, 27]). The idea of Buchberger and Möller has been generalized to the projective space $\mathbb{P}^r$ by careful applications of the Hermite interpolation (see, for example, [19, ch.4] or [20] for a survey about the Hermite interpolation). In this paper, we present an algorithm for computing the Hilbert function of fat points by a technique which is a very natural consequence of the original idea of Buchberger and Möller for simple points. The applied method has been implemented using the object oriented language C++ in a software called Points [11]. Using arithmetic on $K = \mathbb{Z}_p$, where p is a prime, and the NTL library of V. Shoup [29], the software Points provides in any projective space over $K = \mathbb{Z}_p$ the computation of the Hilbert function for fat points and for the projective closure of affine points with given multiplicities. Moreover, it is able to compute a minimal set of homogeneous generators for the ideal of fat points following the method described in [10] and in [4] for simple points. However, for the purpose of this paper we need only to prove that the Hilbert function of the ideal of n general fat points in the plane (with the same multiplicity m) is the maximum possible. Since this is an open condition it is enough to find an example of fat points with integer coordinates that have maximal Hilbert function. This turns out to be a condition on the maximality of the rank of certain matrices, and all the computations can be done over $K = \mathbb{Z}_p$ for a suitable prime p. In practice we have used $p = 32003$, on an Intel Pentium III 686 MHz with 128 MB RAM + 240 MB swap, running Linux (kernel 2.2.14). In section 2 we recall some basic facts and present a summary of the main results for the application of the degeneration technique developed by Ciliberto and Miranda together with a direct computation on fat points. In section 3 we describe the algorithm for computing on fat points and the results of its performance. We have also considered an alternative computational approach to this problem, described briefly below in Remark (3.2); but a direct computation with fat points turns out to be more efficient in the cases we are considering. For a survey of this problem from the point of view of approximation and interpolation theory, the article [3] and the more recent [20] are excellent. For a survey of the problem from an algebraic geometry viewpoint, and its connections with other geometric and algebraic problems, as the Waring problem for forms, see [6], [21] and the references therein.

2 Background

Let us begin by first developing the notation necessary to precisely state the conjecture of Harbourne and Hirschowitz that we will then verify (up through multiplicity 20).

Definition 2.1 Given two linear systems $\mathcal{L} = \mathcal{L}_d(m_1, \dots, m_n)$ and $\mathcal{L}' = \mathcal{L}'_{d'}(m'_1, \dots, m'_n)$ of plane curves, their *extra-intersection number* is $< \mathcal{L}, \mathcal{L}' >= dd' - \sum_i m_i m'_i$. The *self-intersection* of $\mathcal{L}$ is the integer $< \mathcal{L}^2 >=< \mathcal{L}, \mathcal{L} >= d^2 - \sum_i m_i^2$ obtained by extra-intersecting $\mathcal{L}$ with itself.

The reader familiar with algebraic geometry methods will recognize this as the intersection numbers of the corresponding linear systems on the blowup of the plane at the n base points.

Definition 2.2 An irreducible rational curve A, which is a member of a linear system $\mathcal{L} = \mathcal{L}_d(m_1, \dots, m_n)$ and such that the proper transform of A on the blow-up $\tilde{\mathbb{P}}^2$ is smooth, of self-intersection -1, is called a (-1)-*curve.*

Definition 2.3 A linear system $\mathcal{L}$ is (-1)-*special* it is nonempty and there is a (-1)-curve A such that $\mathcal{L} \cdot A \leq -2$.

Conjecture 2.1 (Harbourne-Hirschowitz) *A linear system $\mathcal{L}_d(m_1, \dots, m_n)$ of plane curves with general multiple base points is special if and only if it is (-1)-special.*

It is not hard to see that every (-1)-special system is special [7, Lemma 4.1]; hence the real content of the conjecture is that every special system is (-1)-special [8, The Main Conjecture]. Moreover, the only (-1)-special homogeneous systems $\mathcal{L}_d(m^n)$ occur when $n \leq 8$ [8, Theorem 2.4] and it is known that in this range the conjecture holds. Therefore the conjecture can be reformulated, in the homogeneous case, by saying that all homogeneous systems $\mathcal{L}_d(m^n)$ for $n \geq 9$ are non-special. Note that for every degree d there is a critical number $n_0 = \min\{n | \nu \geq -1\}$ such that the virtual dimension of $\mathcal{L}_d(m^{n_0})$ is positive while the dimension of $\mathcal{L}_d(m^{n_0+1})$ is negative. In addition, if $\mathcal{L}_d(m^n)$ is empty, then $\mathcal{L}_d(m^q)$ is empty for all $q \geq n$; and if $\mathcal{L}_d(m^n)$ is nonempty and non-special, then $\mathcal{L}_d(m^q)$ is nonempty and non-special for all $q \leq n$. Therefore, if one can show that the critical system $\mathcal{L}_d(m^{n_0})$ is non-special and that the system $\mathcal{L}_d(m^{n_0+1})$ is empty, then $\mathcal{L}_d(m^n)$ will be non-special for all n. Hence for fixed multiplicity m, and fixed degree d, we have a priori only two cases to check in order to prove the conjecture. The degeneration technique of Ciliberto and Miranda (used in [7, 8] to prove the conjecture for $m \leq 12$) provides a recursion in the degree d for fixed multiplicity m. As is the case with many such arguments, there are various technical difficulties in applying the recursion for low values of the degree d. Ciliberto and Miranda define the function

$$dlow(\gamma, h) = \Big(\binom{m}{2} + \binom{\gamma+1}{2} + (2h+1)\binom{m+1}{2} - m\gamma - 1\Big)/(m+1-\gamma)$$

and set

$$D(m) = max\{\lfloor (23m+16)/6 \rfloor, \lceil dlow(-1, \lceil (m^2-1)/(3m+4) \rceil) \rceil\}.$$

Ciliberto and Miranda are only able to prove that the recursion always works when the degrees are larger than $D(m)$. Specifically, by applying the degeneration technique, they obtain the following result.

Theorem 2.1 *Fix $m \geq 2$ and let $D = D(m)$ as defined above. Suppose that the conjecture holds for all linear systems $\mathcal{L}_d(m^n)$ with $d < D$. Then the conjecture holds for all linear systems $\mathcal{L}_d(m^n)$.*

Proof. [8, Theorem 4.1].

For low degrees more standard methods based on the theory of Cremona transformations gives the following result.

Proposition 2.1 *Fix m and $d \leq 3m$. Then for all n the conjecture holds for the homogeneous linear system $\mathcal{L}_d(m^n)$.*

Proof. [8, Proposition 5.1].

From the above considerations we see that the conjecture for a fixed m and all d and n will follow if one can show that for all d in the range $[3m+1, D(m)-1]$ the system $\mathcal{L}_d(m^{n_0})$ is non-special and the system $\mathcal{L}_d(m^{n_0+1})$ is empty. For fixed m this then reduces the problem to a finite computation. The degeneration method does not fail for all d in this interval, and can further be used to reduce the number of cases one has to handle by other methods. By their method Ciliberto and Miranda investigated the dimension of $\mathcal{L}_d(m^n)$ up through $m = 12$ and proved the conjecture in this range. Here we go further, reaching the value $m = 20$.

3 Proving the conjecture for $\mathcal{L}_d(m^n)$ up through $m = 20$ by computation on points

Recall that here we are interested in the dimension of the linear system $\mathcal{L} = \mathcal{L}_d(m^n)$ consisting of plane curves of degree d with multiplicity at least m at n general points. As noted above, this is equivalent to computing the Hilbert function of S/I when I is the ideal of fat points in the plane. The method we present below in fact works even in higher-dimensional projective spaces, and we present the algorithm in this more general context. Let $p_1, \dots, p_n$ be n distinct points in $\mathbb{P}^r$ and $\mathcal{P}_i$ the ideal of the point p_i. Recall that these points are called *fat points* with multiplicity at least $m_1, \dots, m_n$ respectively when they are the support of the scheme $\mathbb{X}$ whose saturated ideal is $I = I(\mathbb{X}) = \mathcal{P}_1^{m_1} \cap \dots \cap \mathcal{P}_n^{m_n}$. Note that, if $m = \max\{m_i\}_{i=1,\dots,n}$ is the maximum of the multiplicities, then a homogeneous polynomial f belongs to I if and only if f has degree at least m and its derivatives of order $m_i - 1$ vanish at the point p_i for each $i = 1, \dots, n$. Our aim is to compute the Hilbert function of $\mathbb{X}$ when the fat points are general. Hence we can assume that the points belong to the non empty open set $U = \mathbb{P}^r - V(x_0)$, where $V(x_0)$ is the hyperplane of points whose first coordinate is zero. This choice allows us to take points in the affine space $\mathbb{A}^r$ which is identified with U. A generalization of the method of Buchberger and Möller to affine points with differential conditions has been already described in [22, 2]. We implemented an ad hoc algorithm for our purposes and here we describe how we tackle the computation of the Hilbert function of $\mathbb{X}$ both in projective and in affine space. Recall that if S_d is the set of all homogeneous polynomials of degree d in $S = K[x_0, \dots, x_r]$ and I_d is the set of all homogeneous polynomials of degree d of I, then $H(S/I, d) := dim_K(S_d/I_d)$ is the Hilbert function of $\mathbb{X}$ (or of S/I). Since I is saturated and S/I is a ring of dimension 1, then S/I is a Cohen-Macaulay ring. Therefore the Hilbert function of $\mathbb{X}$ is strictly increasing until it assumes the value of the Hilbert polynomial $HP = \sum_{i=1}^{n} \binom{m_i-1+r}{r}$ of S/I. If we want to compute the Hilbert function of $\mathbb{X}$ we need to determine the dimension of the K-vector space S_d/I_d. Let $\mathcal{T} = \{x_0^{\alpha_0} \dots x_r^{\alpha_n} | (\alpha_0, \dots, \alpha_r) \in \mathbb{N}\}$ be the monoid of the terms of S and suppose that $\mathcal{T}$ is ordered with respect to a graded term order $<$, i.e. a graded well-ordering of monoids. To interpolate the fat points $p_1, \dots, p_n$, for each degree d we consider the matrix G_d^n such that the j-th column is the vector of the evaluations at the given points p_i of the derivatives of order $m_i - 1$ of the j-th term (in the fixed term order) T of degree d. Let $T_1^{(d)} < \dots < T_{q_t}^{(d)}$ be the terms of degree d corresponding to the first columns of G_d^n that form a maximal set of linearly independent columns. We say that such terms are *independent* and that the others are *dependent*. Now, suppose that the j-th term T is dependent and let $a_1, \dots, a_{q_t}$ be the scalars of the linear combination of the columns corresponding to $T_1^{(d)}, \dots, T_{q_t}^{(d)}$ which is equal to the column of T. With a notation similar to that introduced in [22], we consider the following polynomial

$$b(T) = T - \sum_{i=1}^{q_t} a_i T_i^{(d)}. \tag{1}$$

For each dependent term T we obtain the scalars $a_1, \dots, a_{q_t}$ by computing the reduced row-echelon form of the matrix G_d^n. Note that, for each i with $a_i \neq 0$, it is $T_i^{(d)} < T$.

Let $\mathcal{S}_d$ be the set of all polynomials of type $b(T)$ of degree d computed by the above method. Of course $\mathcal{S}_d$ is a K-basis of the K-vector space I_d. Let $\sigma = min\{d \in \mathbb{N} | H(S/I, d) = H(S/I, d-1)\}$ and recall that S/I is a Cohen-Macaulay ring of dimension 1. If f is a polynomial, denote its leading term by $LT(f)$; define $LT(I) =< LT(f) | f \in I >$ to be the ideal generated by the leading terms of the polynomials of I. Moreover, if B is a set of non null polynomials, denote by $LT(B) = \{LT(f) | f \in B\}$ the set of the leading terms of the polynomials of B. Recall that a Gröbner basis of the ideal I with respect to a fixed term ordering is a subset G of non null polynomials of I such that $LT(G) = LT(I)$. Moreover, a Gröbner basis is always a set of generators of I.

Proposition 3.1 *$\mathcal{S} = \cup_{d=1}^{\sigma} \mathcal{S}_d$ is a set of generators of I and consists of the polynomials of degree $\leq \sigma$ of a Gröbner basis for I with respect to the set term ordering $<$.*

Proof. Since S/I is a Cohen-Macaulay ring of dimension 1, then σ is the regularity of Castelnuovo-Mumford of I and so $\mathcal{S} = \cup_{d=1}^{\sigma} \mathcal{S}_d$ is a set of generators for I. Therefore it is sufficient to show that $LT(I)_d = LT(\mathcal{S})_d$. Since $\mathcal{S} \subseteq I$, obviously $LT(\mathcal{S})_d \subseteq LT(I)_d$. Let f be a polynomial of I_d, where d is lower than or equal to σ. By construction, all polynomials of $\mathcal{S}_d$ have different leading terms. Furthermore, each leading term does not appear (i.e. the corresponding coefficient vanishes) in the polynomials where it is not the leading term. Thus, since every polynomial f of I_d is a linear combination of polynomials of $\mathcal{S}_d$, $LT(f)$ must be the leading term of a polynomial of $\mathcal{S}$.

Proposition 3.2 *If T is a dependent term of degree d then every term $T\bar{T}$ multiple of T is a dependent term too and the polynomial $b(T\bar{T})$ depends on $\mathcal{S} - b(T\bar{T})$.*

Proof. With the same notation set before, by the hypothesis the vector of the evaluations of T depends on the vectors of the evaluations of $T_1^{(d)}, \dots, T_{q_t}^{(d)}$ and so the vector of the evaluations of $T\bar{T}$ depends on the vectors of the evaluations of $T_1^{(d)}\bar{T}, \dots, T_{q_t}^{(d)}\bar{T}$. Thus $T\bar{T}$ is a dependent term and the polynomial $b(T\bar{T})$ exists. By construction, the leading term of the polynomial $p_1 = b(T\bar{T}) - \bar{T}b(T)$ is lower than $T\bar{T}$. Since p_1 belongs to the ideal I and since $\mathcal{S}$ is a Gröbner basis, there is a polynomial p of $\mathcal{S}$ such that $T(p) = T(p_1)$. Let $p_2 = p_1 - p$ and note that $T(p_2) < T(p_1)$. Since p_2 belongs to I, we can repeat for p_2 the same procedure applied to p_1. If we repeat this "rewriting procedure" we obtain a polynomial p_k which is null because a term ordering is a well-ordering.

By Proposition 3.2, when we compute the matrix G_d^n of the evaluations of the terms of degree d we can eliminate each term $T\bar{T}$ which is multiple of a dependent term T. Hence, we obtain the submatrix $\bar{G}_d^n$ of G_d^n whose columns correspond to the terms of degree d that are divided *only* by the independent terms of degree $d-1$. At the same time, we avoid to construct each corresponding polynomial $b(T\bar{T})$ obtaining a subset G' of $\mathcal{S}$ which consists exactly of the polynomials of degree $\leq \sigma$ of the reduced Gröbner basis of I with respect to the set term ordering. It is important to note that $H(S/I,d) = rank(G^n_d) = rank(\bar{G}^n_d)$ is equal to the number of independent terms of degree d. The following algorithm is based on the above observations.

Algorithm I

Input: n distinct points of $\mathbb{P}^r$ and their multiplicities $m_1, \dots, m_n$. **Output:** The Hilbert function of the projective scheme $\mathbb{X}$ of the given fat points.

begin

1. set $HP = \sum_{i=1}^n \binom{m_i-1+r}{r}$, $m = max\{m_i\}_{i=1,\dots,n}$ and $d = m-1$;

2. **for** $i = 0, \dots, m-1$ **do** $H(S/I,d) = H(S,d)$ **endfor**

3. **while** $H(S/I,d) < HP$ **do**

- **3.1** set $d = d+1$;
- **3.2** computation of the terms of degree d which are divided only by the independent terms of degree $d-1$;
- **3.3** computation of $\bar{G}^n_d$;
- **3.4** set $H(S/I,d) = rank(\bar{G}^n_d)$;

endwhile

end

For our purposes it is enough to take input points which belong to the already defined affine open set U, so that the first projective coordinate of the points is always 1. Hence the above algorithm can be modified as it follows. Let $\bar{H}^n_d$ be the transpose matrix of $\bar{G}^n_d$. In this case the independent terms are never divided by x_0.

Algorithm II

Input: n distinct points of $\mathbb{P}^r$ belonging to U and their multiplicities $m_1, \dots, m_n$. **Output:** The Hilbert function of the projective scheme $\mathbb{X}$ of the given fat points.

begin

1. set $HP = \sum_{i=1}^n \binom{m_i-1+r}{r}$, $m = \max\{m_i\}_{i=1,\dots,n}$ and $d = m-1$;

2. **for** $i = 0, \dots, m-1$ **do** $H(S/I,d) = H(S,d)$ **endfor**

3. computation and Gauss reduction of $\bar{H}^n_d$

4. **while** $H(S/I,d) < HP$ **do**

4.1 set $d = d + 1$;

4.2 computation of the set T_d of terms of degree d in the n variables $x_1, \ldots, x_n$ which are divided only by the independent terms in n variables of degree $\leq d - 1$;

4.3 computation of the rows of $\bar{H}_d^n$ corresponding to the terms of degree d that have been conputed in step 4.2; Gauss reduction of these rows with respected to the already reduced rows of $\bar{H}_{d-1}^n$;

4.4 set $H(S/I, d) = rank(\bar{H}_d^n)$;

endwhile

end

We have

$$H(S/I, d) \leq \min\{H(S, d), HP\} = \min\{\binom{d+r}{r}, \sum_{i=1}^{n} \binom{m_i - 1 + r}{r}\}$$

and $H(S/I, d) = min\{H(S, d), HP\}$ for some d if and only if the linear system $\mathcal{L} = \mathcal{L}(m_1, \ldots, m_n)$ has the expected dimension for that d. This last is an open condition [21] and is equivalent to the maximality of $rank(\bar{H}_d^n)$. Now consider a linear system $\mathcal{L} = \mathcal{L}(m_1, \ldots, m_n)$ whose points $p_1, \ldots, p_n$ have integer coordinates and let $\overline{p}_1, \ldots, \overline{p}_n$ be the points whose coordinates are the coordinates of $p_1, \ldots, p_n$ modulo a prime p. Let $\overline{I} = \overline{\mathcal{P}}_1^{m_1} \cap \ldots \cap \overline{\mathcal{P}}_n^{m_n} \subset \overline{S} = \mathbb{Z}_p[X_0, ..., X_r]$ where $\overline{\mathcal{P}}_i$ is the ideal of the point $\overline{p}_i$. By the previous considerations if $H(\overline{S}/\overline{I}, d) = \min\{\binom{d+r}{r}, \sum_{i=1}^{n} \binom{m_i-1+r}{r}\}$ then a generic linear system $\mathcal{L} = \mathcal{L}(m_1, \ldots, m_n)$ over the complex numbers has the expected dimension. So, since to have the expected dimension is an open condition, we implemented Algorithm II for points with randomly generated coordinates over $K = \mathbb{Z}_p$. In practice it turned out that for all computations it was enough to consider $p = 32003$.

Remark 3.1 The described algorithms consist essentially of iterated Gaussian eliminations. Hence, for evaluating their computational cost it is enough to look at the dimensions of the matrices to be reduced. To this aim note that the number of columns of the matrix $\bar{G}_d^n$ is no higher than $HP{\cdot}(r + 1)$ (where recall that $HP = \sum_{i=1}^{n} \binom{m_i-1+r}{r})$ because the number of independent terms of degree d is lower than HP for each d. Moreover, note that the number of rows is equal to $HP{\cdot}n$ because there are HP rows for each point. We consider the multiplication as the significative operation and so the computational cost of the rank of $\bar{G}_d^n$ is of order $O(HP^3{\cdot}r)$. Since it is $d < HP$, we see that Algorithm I is an $O(HP^4{\cdot}r)$ algorithm. For fat points in the open subset U of the projective space we have to perform a Gaussian reduction of only one matrix $\bar{H}_d^n$, with $d = min\{t|H(S/I, d) = HP\} < HP$; hence the computational cost of Algorithm II is of order $O(HP^3{\cdot}r)$. In particular, if the points are in the plane ($r = 2$) and of the same multiplicity m, the computational cost of Algorithm I becomes of order $O(n^4m^8)$ and that of Algorithm II becomes of order $O(n^3m^6)$.

As shown in [8] it is possible to write a computer program that for a fixed positive integer m gives all the finitely many triples (d, n, m) for which the degeneration method of Ciliberto and Miranda fails to compute the dimension of the linear system of plane curves of degree d having n points of multiplicity m, and therefore fails to give an answer to Conjecture 2.1. In the following tables we list such exceptional triples (d, n, m) with m in the range $13 \leq m \leq 20$, with d in the corresponding range $3m + 1 \leq d \leq D(m) - 1$, and with n either given by the critical value n_0 or by $n_0 + 1$. The corresponding virtual dimension ν is also listed. The cases for which $m \leq 12$ have been treated by geometric arguments in [8]. The intervals $[3m+1, D(m)-1]$ for $13 \leq m \leq 20$ are given in the table below:

m	13	14	15	16	17	18	19	20
3m+1	40	43	46	49	52	55	58	61
D(m)-1	60	78	84	100	112	119	144	193

d	n	m	ν	d	n	m	ν	d	n	m	ν	d	n	m	ν
40	10	13	-50	52	13	14	65	49	10	16	-86	52	10	17	-100
41	10	13	-8	65	21	14	5	50	10	16	-35	53	10	17	-46
42	10	13	35	78	30	14	9	51	10	16	17	54	10	17	9
43	10	13	79	46	10	15	-73	52	10	16	70	55	10	17	65
45	12	13	-12	47	10	15	-25	53	10	16	124	56	10	17	122
46	12	13	35	48	10	15	24	54	11	16	43	57	11	17	27
47	13	13	-8	49	10	15	74	56	12	16	20	59	12	17	-7
48	13	13	41	50	11	15	5	57	12	16	78	60	12	17	54
59	20	13	9	52	12	15	-10	58	13	16	1	61	13	17	-37
43	10	14	-61	53	12	15	44	59	13	16	61	62	13	17	26
44	10	14	-16	54	13	15	-21	60	14	16	-14	63	13	17	90
45	10	14	30	55	13	15	35	60	13	16	122	64	14	17	2
46	10	14	77	56	13	15	92	73	20	16	54	75	19	17	18
47	11	14	20	66	19	15	-3	74	21	16	-7	77	20	17	20
49	12	14	14	68	20	15	14	75	21	16	69	79	21	17	26
51	13	14	12	70	21	15	35	89	30	16	14	80	21	17	107

d	n	m	ν	d	n	m	ν	d	n	m	ν	d	n	m	ν
96	31	17	9	100	30	18	20	84	19	19	44	71	12	20	107
112	42	17	14	102	31	18	54	86	20	19	27	72	13	20	-30
55	10	18	-115	58	10	19	-131	88	21	19	14	73	13	20	44
56	10	18	-58	59	10	19	-71	89	21	19	104	74	13	20	119
57	10	18	0	60	10	19	-10	90	22	19	5	75	14	20	-15
58	10	18	59	61	10	19	52	107	31	19	-5	75	13	20	195
59	10	18	119	62	10	19	115	108	31	19	104	76	14	20	62
60	11	18	9	63	11	19	-11	125	42	19	20	88	19	20	14
61	11	18	71	63	10	19	179	61	10	20	-148	90	20	20	-15
63	12	18	27	64	11	19	54	62	10	20	-85	91	20	20	77
64	12	18	92	66	12	19	-3	63	10	20	-21	92	21	20	-40
65	13	18	-13	67	12	19	65	64	10	20	44	93	21	20	54
66	13	18	54	68	12	19	134	65	10	20	110	94	21	20	149
67	13	18	122	69	13	19	14	66	11	20	-33	95	22	20	35
68	14	18	20	70	13	19	85	66	10	20	177	109	29	20	14
81	20	18	-18	71	14	19	-33	67	11	20	35	111	30	20	27
82	20	18	65	71	13	19	157	68	11	20	104	113	31	20	44
84	21	18	63	72	14	19	40	70	12	20	35	133	43	20	14

Using the software Points we have investigated the above exceptional cases and we have found that in these cases the Hilbert function S/I at the degree d is always maximal (recall that I is the ideal of n general fat points of multiplicity m). Thus we complete the proof of the Harbourne-Hirschowitz conjecture up to $m = 20$. Recall that the Hilbert function of S/I is strictly increasing until it assumes constantly its maximum value HP. Since our software Points computes the Hilbert function of S/I until this function reachs the maximum value HP, we prove the cases related to the same number n of points by only one performance of Points. So in the following tables, for each m with $13 \leq m \leq 20$, we collect the exceptional couples (d, n) and list the minimum degree α of the generators of the corresponding ideal I, the Hilbert polynomial HP, the timing "Time" of the computation in seconds and the memory space "size" used during the computation in megabytes. Note that α is the minimum degree at which the virtual dimension is non negative.

Multiplicity m=13

(d,n)	Time	α	HP	size
(40,10) (41,10) (42,10) (43,10)	47.21	42	910	2.336
(45,12) (46,12)	82.13	46	1092	3.052
(47,13) (48,13)	104.55	48	1183	3.460
(59,20)	384.07	59	1820	7.216

Multiplicity m=14

(d,n)	Time	α	HP	size
(43,10) (44,10) (45,10) (46,10)	71.93	45	1050	2.876
(47,11)	96.07	47	1155	3.332
(49,12)	125.17	49	1260	3.828
(51,13) (52,13)	159.79	51	1365	4.372
(65,21)	679.38	65	2205	10.252
(78,30)	1994.65	78	3150	20.160

Multiplicity m=15

(d,n)	Time	α	HP	size
(46,10) (47,10) (48,10) (49,10)	106.83	48	1200	3.540
(50,11)	142.27	50	1320	4.784
(52,12) (53,12)	185.16	53	1440	5.488
(54,13) (55,13) (56,13)	236.55	55	1560	5.492
(66,19)	747.30	67	2280	10.912
(68,20)	870.72	68	2400	12.012
(70,21)	1008.43	70	2520	13.168

Multiplicity m=16

(d,n)	Time	α	HP	size
(49,10) (50,10) (51,10) (52,10) (53,10)	153.67	51	1360	4.344
(54,11)	205.57	54	1496	51.08
(56,12) (57,12)	268.64	56	1632	5.944
(58,13) (59,13) (60,13)	342.39	58	1768	6.848
(60,14)	431.12	61	1904	7.828
(73,20)	1262.23	73	2720	15.220
(74,21) (75,21)	1464.37	75	2856	16.708
(89,30)	4302.29	89	4080	33.324

Multiplicity m=17

(d,n)	Time	α	HP	size
(52,10) (53,10) (54,10) (55,10) (56,10)	217.55	54	1530	5.312
(57,11)	291.45	57	1683	6.276
(59,12) (60,12)	379.46	60	1836	7.332
(61,13) (62,13) (63,13)	484.70	62	1989	8.476
(64,14)	607.76	64	2142	9.716
(75,19)	1534.83	75	2907	17.284
(77,20)	1792.55	77	3060	19.068
(79,21) (80,21)	2079.31	79	3213	20.948
(96,31)	6764.49	96	4743	44.764
(112,42)	16955.24	112	6426	82.176

Multiplicity m=18

(d,n)	Time	α	HP	size
(55,10) (56,10) (57,10) (58,10) (59,10)	301.84	57	1710	6.456
(60,11) (61,11)	404.39	60	1881	7.660
(63,12) (64,12)	527.84	63	2052	8.976
(65,13) (66,13) (67,13)	673.66	66	2223	10.412
(68,14)	844.88	68	2394	11.956
(81,20) (82,20)	2498.38	82	3420	24.288
(84,21)	2899.71	84	3591	25.984
(100,30)	8574.12	100	5130	52.236
(102,31)	9456.25	102	5301	55.688

Multiplicity m=19

(d,n)	Time	α	HP	size
(58,10)	411.48	61	1900	7.800
(59,10)				
(60,10)				
(61,10)				
(62,10)				
(63,10)				
(63,11)	551.84	64	2090	9.284
(64,11)				
(66,12)	720.51	67	2280	10.912
(67,12)				
(68,12)				
(69,13)	921.17	69	2470	12.680
(70,13)				
(71,13)				
(71,14)	1154.14	72	2660	14.592
(72,14)				
(84,19)	2933.94	84	3610	26.248
(86,20)	3424.97	86	3800	29.008
(88,21)	3978.13	88	3990	31.900
(89,21)				
(90,22)	4974	90	4180	34.940
(107,31)	12966.87	108	5890	68.616
(108,31)				
(125,42)	58316.28	125	7980	121

Multiplicity m=20

(d,n)	Time	α	HP	size
(61,10)	564.73	64	2100	9.368
(62,10)				
(63,10)				
(64,10)				
(65,10)				
(66,10)				
(66,11)	758.32	67	2310	11.180
(67,11)				
(68,11)				
(70,12)	976.20	70	2520	13.168
(71,12)				
(72,13)	1240.91	73	2730	15.332
(73,13)				
(74,13)				
(75,13)				
(75,14)	1555.49	76	2940	17.664
(76,14)				
(88,19)	3958.99	88	3990	31.900
(90,20)	4649.01	91	4200	35.268
(91,20)				
(92,21)	5367.77	93	4410	38.804
(93,21)				
(94,21)				
(95,22)	6167.79	95	4620	42.516
(109,29)	14292.99	109	6090	73.300
(111,30)	15803.59	111	6300	78.348
(113,31)	17506.68	113	6510	83.652
(133,43)	158147.99	133	9030	155

Remark 3.2 Applying the theory of linear inverse systems (see [12] for a brief review and, for example, [17] for a deeper study), the Hilbert function of S/I at a degree d is equal to the Hilbert function at the same degree d of an ideal $(L_1^{d-m_1+1}, \dots, L_n^{d-m_n+1})$ generated by powers of n general linear forms from S_1. So, for example, if $m_i = m$, $r = 2$ and $n \leq 3$, setting $L_1 = x_0$, $L_2 = x_1$ and $L_3 = x_2$, it is not difficult to single out the cases in which the linear systems $\mathcal{L}_d(m^n)$ are special. So, instead of computing with fat points, one could try to compute the Hilbert function of the ideal $(L_1^{d-m_1+1}, \dots, L_n^{d-m_n+1})$. We did this by CoCoA [5], a computer algebra software package that is specialized for the computation of the Hilbert function. But it turns out that this approach becomes less efficient than the computation with fat points when the values of m and n increase.

References

1. J. Abbott, A. Bigatti, M. Kreuzer, L. Robbiano (2000). Computing Ideals of Points. *J. Symbolic Computation* **30**, no.4, 351-356.

2. J. Abbott, M. Kreuzer, L. Robbiano (2000). Computing Zero-dimensional Schemes. Available at `http://cocoa.dima.unige.it/research/publications.html`.

3. A.A. Akopyan and A. A. Saakyan (1994). Multivariate splines and polynomial interpolation. *Russian mathematical surveys* **48**, no. 5 (1994), 1–72.

4. G. Albano, F. Cioffi, F. Orecchia, I. Ramella (2000). Minimally generating ideals of rational parametric curves in polynomial time. *J. Symbolic Computation* **30**, no. 2, 137-149.

5. A. Capani, G. Niesi, L. Robbiano(2000). CoCoA, a system for doing Computations in Commutative Algebra. Available via anonymous ftp from: `cocoa.dima.unige.it`, edition 4.0.

6. C. Ciliberto (2000). Geometric Aspects of Polynomial Interpolation in More Variables and Waring's problem. Proceeding of the 3ECM, Barcelona. In press. Progress in Mathematics, Birkhäuser.

7. C. Ciliberto, R. Miranda (1998). Degenerations of Planar Linear Systems. *Journal reine Angew. Math.*, **501**, 191-220.

8. C. Ciliberto, R. Miranda (2000). Linear systems of plane curves with base points of equal multiplicity. *Trans. Amer. Math. Soc.* **352**, 4037-4050.

9. C. Ciliberto, R. Miranda (2001). The Segre and Harbourne-Hirschowitz Conjectures. to appear.

10. F. Cioffi (1999). Minimally generating ideals of points in polynomial time using linear algebra. *Ricerche di Matematica* **XLVIII**, Fasc. 1, 55-63.

11. F. Cioffi, F. Orecchia, I. Ramella.(2000) Points (software for computations on points) Available at `http://cds.unina.it/~orecchia/gruppo/EPoints.html`.

12. A. V. Geramita (1996). Inverse Systems of Fat Points: Waring's Problem, Secant Varieties of Veronese Varieties and Parameter Spaces for Gorenstein Ideals. The Curves Seminar at Queen's, Vol X (Queen's Papers in Pure and Applied Mathematics no. 102), 2-129.

13. A. V. Geramita, F. Orecchia (1982). Minimally generating ideals defining certain tangent cones. *J. Algebra* **78**, no. 1, 36–57.

14. B. Harbourne (1986). The Geometry of rational surfaces and Hilbert functions of points in the plane. *Can. Math. Soc. Conf. Pro.*, **6** (1986), 95-11.

15. A. Hirschowitz (1985), La methode d'Horance pour l'interpolation a plusieurs variables. *Man. Math.* **50**, 337-388

16. A. Hirschowitz (1989), Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles generiques. *J. reine angew. Math.* **397**, 208-213.

17. A. Iarrobino (1995). Inverse System of a Symbolic power II. The Waring Problem for Forms. *Journal of Algebra* **174**, 1091-1110.

18. G. G. Lorentz, R. A. Lorentz (1990). Bivariate Hermite Interpolation and Application to Algebraic Geometry. *Numer. Math.* **57**, 669-680.

19. R. A. Lorentz (1992). Multivariate Birkhoff Interpolation. LNM 1516, Springer Verlag,1992.

20. R. A. Lorentz (2000). Multivariate Hermite interpolation by algebraic polynomials: a survey. Numerical analysis 2000, Vol. II: Interpolation and extrapolation. *J. Comput. Appl. Math.* **122**, no. 1-2, 167–201.

21. R. Miranda (1999). Linear systems of plane curves. *Notices of the AMS*, Volume **46**, Number 2, 192-202.,

22. M. G. Marinari, H. M. Möller, T. Mora (1993), Gröbner Bases of Ideals Defined by Functionals with an Application to Ideals of Projective Points. *AAECC*, **4**, 103-145, 1993.

23. H. M. Möller, B. Buchberger (1982), The construction of multivariate polynomials with preassigned zeros. EUROCAM 82, L.N.C.S. 144, Springer-Verlag, 1982.

24. F. Orecchia, (1991). A polynomial algorithm for computing the conductor of points and of a curve with reduced tangent cone. *Queen's Papers in Pure and Applied Mathematics no* **88**, *The Curves Seminar at Queen's* **VIII**, Queen's University, Kingston, Ontario,Exp. G, 8 pp..

25. Ramella, I. (1989). An algorithmic approach to ideal generation of points. *Rend. Accad. Sci. Fis. Mat.*, **56**, 72–81.

26. Ramella, I. (1990). Algoritmi di Computer Algebra relativi agli ideali di punti dello spazio proiettivo. Tesi di Dottorato di Ricerca, Preprint n. 30, 1990, Dip. di Mat. e Applic. "R. Caccioppoli", Univ. di Napoli "Federico II".

27. Ramella, I. (1994). Ideals of points in generic position: a polynomial algorithm for computing set of generators. *Ricerche di Matematica*, **43**, 205–217.

28. Z. Ran (1989). Enumerative geometry of singular plane curves. *Invent. Math* **97**, 447-465.

29. V. Shoup (2000). NTL: a Library for doing Number Theory. Open source software distributed under the GNU General Public License and available at `http://www.shoup. net/ntl`.