ANACAPRI LECTURES ON DEGENERATIONS OF SURFACES

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Introduction

These notes are a summary of four lectures given at the summer school held in Anacapri in June of 2004. The subject was degenerations of surfaces, and the intention of the talks was simply to give an introduction to very basic facts, an overview of degenerations of $K3$ and abelian surfaces, some elementary constructions of degenerations of rational surfaces, and an indication of possible applications.

1. Lecture One

We begin with the basic definition of a degeneration.

Definition 1.1. A *one-dimensional degeneration* is a flat proper map

$$\pi : \mathcal{X} \to \Delta$$

from a smooth $(n+1)$-fold $\mathcal{X}$ to the disc Δ such that the general fiber $X_t = \pi^{-1}(t)$ for $t \neq 0$ is smooth and the special fiber X_0 is a divisor

$$X_0 = \sum_i n_i V_i$$

where the components V_i meet each other in local normal crossings. The degeneration is *reduced* if all the multiplicities n_i are equal to one: $X_0 = \sum_i V_i$. The degeneration has *global normal crossings* if each V_i is smooth and for each i and j, V_i meets V_j transversally in a smooth irreducible curve.

Given a one-dimensional degeneration, one can always achieve a reduced global normal crossings degeneration after a possible base change and birational modifications; this is semistable reduction.

Example 1.2. Take a curve C_0 on a surface X, which moves in a positive-dimensional linear system. Choose a pencil containing C_0 which has no fixed components. Blow up the base locus of the pencil to a surface $\tilde{X}$, which then comes equipped with a map $\pi : \tilde{X} \to \mathbb{P}^1$. Choose coordinates such that $\pi^{-1}(0)$ is the transform X_0 of C_0. (One may need to blow up further to achieve local normal crossings for X_0.) Restricting π to a neighborhood of 0 gives a one-dimensional degeneration of the general member of the pencil, to X_0.

A *Lefshetz pencil* is an example of the above type of degeneration, where C_0 has only nodes as singularities, all base points are simple, and none of the base points occur at the nodes of C_0. This gives a degeneration of a smooth curve to the nodal curve C_0: under these hypotheses the C_0 is isomorphic to its transform on the resolution of the pencil.

Example 1.3. Apply the previous construction to the case where C_0 is a reducible cubic, consisting of a double line and another line: $C_0 = 2\ell + m$ Take a general cubic curve (meeting the two lines transversally in three points each, not the intersection point $\ell \cap m$) and resolve the pencil. The tranform X_0 in this case consists of five smooth rational curves:

$$X_0 = 2L + M_1 + M_2 + M_3 + M_4$$

where L is the transform of ℓ, M_1 is the transform of m, each M_i meets L in one point, and the M_i's are disjoint. All five curves have self-intersection equal to -2. The general fiber is isomorphic to a smooth cubic curve in the pencil, and is therefore elliptic. This is a degeneration of elliptic curves; in Kodaira's notation it is of type I_0^*.

Example 1.4. Apply the pencil construction where C_0 is a triple line in the plane, and the general member of the pencil meets C_0 transversally in three points. The result is a special fiber

$$X_0 = 3L_0 + 2M_1 + 2M_2 + 2M_3 + N_1 + N_2 + N_3$$

where L_0 is the transform of C_0, L_0 meets each M_i in one point, M_i meets N_i in one point, and all other intersections are empty. All seven curves have self-intersection equal to -2. This degeneration of elliptic curves is of type IV^* in Kodaira's notation.

Suppose now we return to the general setting, with a degeneration whose special fiber is

$$X_0 = \sum_i n_i V_i.$$

Denote by C_{ij} the double curve $V_i \cap V_j$, considered as a curve on V_i. Note that $C_{ij} = C_{ji}$, but are thought of as curves on different surfaces. We call a point of the triple intersection $T_{ijk} = V_i \cap V_j \cap V_k$ a *triple point* of the special fiber.

Proposition 1.5. *With the above notation, we have:*

(a) *The normal bundle of V_i in $\mathcal{X}$ satisfies*

$$N_{V_i/\mathcal{X}}^{n_i} = \mathcal{O}_{V_i}(-\sum_{j \neq i} n_j C_{ij}).$$

(b) *If $\dim(X_t) = 2$, then*

$$V_i^2 V_j = C_{ji}^2$$

where the self-intersection is taken on the surface V_j.

(c) *(The Triple Point Formula) If $\dim(X_t) = 2$,*

$$n_j C_{ij}^2 + n_i C_{ji}^2 = -\sum_{k \neq i,j} n_k |T_{ijk}|.$$

Proof: One first remarks that the general fiber X_t is numerically equivalent to the special fiber X_0. After noting that the normal bundle $N_{V_i/\mathcal{X}}$ on V_i is associated to the divisor $V_i|_{V_i}$, we have that

$$\begin{aligned} 0 = V_i X_t = V_i X_0 &= \sum_j n_j V_i V_j \\ &= n_i V_i|_{V_i} + \sum_{j \neq i} n_j C_{ji} \end{aligned}$$

which proves (a). The second statement follows because restriction commutes with intersection: the quantity $V_i^2 V_j$ can be thought of as taking V_i, intersecting with itself, and restricting to V_j, while the quantity C_{ji}^2 is taking V_i, first restricting to V_j, and then taking the self-intersection.

The Triple Point Formula is obtained by computing $0 = V_i V_j X_0$ and re-arranging as in (a).
QED

We may use some of these considerations to prove that any degeneration of $\mathbb{P}^1$ is trivial, in the following sense.

Proposition 1.6. *Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of $\mathbb{P}^1$. Then π is a blowup of a product $\mathbb{P}^1 \times \Delta$.*

Proof: Write the special fiber of the degeneration as $X_0 = \sum_{i=1}^{\ell} n_i C_i$. If $\ell = 1$, it is easy to see that $n_1 = 1$ and C_1 is smooth and rational; it is therefore a product surface by Tsen's Theorem. Suppose on the contrary that $\ell \geq 2$. Our goal is to show that at least one of the components is a (-1)-curve, and can be blown down, so that we may proceed by induction on ℓ. Note that each C_j^2 must be strictly negative, since

$$0 = C_j X_t = C_j X_0 = n_j C_j^2 + \sum_{i \neq j} n_i C_i C_j$$

and the sum must be strictly positive, since the configuration must be connected. Now since the general fiber is smooth and rational, we have

$$-2 = (K_{\mathcal{X}} + X_t) \cdot X_t = K_{\mathcal{X}} \cdot X_t = K_{\mathcal{X}} \cdot X_0 = \sum_i n_i K_{\mathcal{X}} \cdot C_i = \sum_i n_i [2p_a(C_i) - 2 - C_i^2].$$

Suppose that no C_i is a (-1)-curve. In this case, if any component C_i has arithmetic genus 0, then it must be the case that $C_i^2 \leq -2$; hence the quantity $2p_a(C_i) - 2 - C_i^2$ must be non-negative. If a component has strictly positive arithmetic genus, then this quantity is strictly positive. In any case it is not possible that the sum above equal -2. This contradiction proves that at least one of the components must be a (-1)-curve, and the argument concludes.
QED

The most famous and useful classification result in the theory of degenerations is the list of Kodaira for degenerations of elliptic curves; see [Ko].

Proposition 1.7. *Let $\pi : \mathcal{X} \to \Delta$ be a flat proper map such that $\mathcal{X}$ is smooth, the general fiber X_t is an elliptic curve, and the special fiber has no (-1)-curves as components. Then the special fiber X_0 is either:*

- (mI_0) *a smooth elliptic curve with multiplicity m*
- (mI_1) *a nodal rational curve with multiplicity m*
- (mI_2) *two smooth rational curves meeting transversally at two points, each with multiplicity m*
- (mI_N) *a cycle of N smooth rational curves meeting transversally, with multiplicity m*
- (I_N^*) *$N+5$ smooth rational curves; $N+1$ of multiplicity 2 meeting in a chain, and four of multiplicity one meeting two each at the end components of the chain (if $N = 0$, all four meet the single component)*
- (II) *a cuspical rational curve*
- (III) *two smooth rational curves meeting to order two at one point*
- (IV) *three smooth rational curves all meeting (transversally) at one point*

(IV^*) $3C_0 + 2D_1 + 2E_1 + 2F_1 + D_2 + E_2 + F_2$, *meeting as in the extended Dynkin diagram of type* $\tilde{(E_6)}$.

(III^*) $4C_0 + 3D_1 + 2E_1 + 3F_1 + 2D_2 + 2F_2 + D_3 + F_3$, *meeting as in the extended Dynkin diagram of type* $\tilde{(E_7)}$.

(IV^*) $6C_0 + 5D_1 + 3E_1 + 4F_1 + 4D_2 + 2F_2 + 3D_3 + 2D_4 + D_5$, *meeting as in the extended Dynkin diagram of type* $\tilde{(E_8)}$.

See also the monograph [M2].

We have encountered, in the case of degenerations of curves, that blowups in the special fiber are always possible, leading to the notion of a minimal model for a degeneration of curves: one in which there are no (-1)-curves occuring as components of the special fiber. In the case of degeneration of surfaces, birational modifications are much more complicated. The most common types are:

- (Blowup a point): take a point $p \in X_0$ and blow it up in the smooth total space $\mathcal{X}$, obtaining an exceptional divisor which is isomorphic to a $\mathbb{P}^2$ with normal bundle of degree -1
- (Blow up a smooth curve): take a smooth curve C in one of the components V_i of X_0, which meets the double curves on V_i transversally, and blow it up, obtaining an exceptional divisor which is a minimal ruled surface R over C; its normal bundle will meet the fiber of the ruling -1 times
- (Blow down to a point): If a component V_i of X_0 is isomorphic to a $\mathbb{P}^2$ and has normal bundle of degree -1, blow it down to a smooth point
- (Blow down to a curve): if a component V_i of X_0 is isomorphic to a minimal ruled surface R and has normal bundle that meets the fiber of R -1 times, blow it down to a smooth curve
- (Throwing (-1)-curve): if a component V_i of X_0 contains a (-1)-curve that meets a single double curve C_{ij} transversally in one point, blow it up (obtaining a ruled surface which must be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$), and blow down the ruled surface via the other ruling, effectively contracting the original (-1)-curve on V_i and creating one on V_j
- (Flipping a (-1) double curve): If a double curve $C = V_i \cap V_j$ satisfies $C_{ij}^2 = C_{ji}^2 = -1$, blow it up (obtaining a ruled surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$), and blow down the ruled surface via the other ruling; this contracts C on V_i and V_j and blows up the other two surfaces that meet C.

See [BGOD] for more details and applications.

Let us close with two interesting and useful examples.

Example 1.8. Recall the notation that $\mathbb{F}_n$ is the minimal rational ruled surface with a section B of self-intersection $-n$. Let $\mathcal{X} = \mathbb{F}_n \times \Delta$ be the trivial family of rational ruled surfaces $\mathbb{F}_n$. On $\mathbb{F}_n$, there are sections $S_k \equiv B + (n+k)F$ which meet B k times and have self-intersection $n + 2k$. In the three-fold $\mathcal{X}$, blow up a section S_k in the central fiber. We obtain a rational ruled surface R, meeting the original $\mathbb{F}_n$ along the section $C = S_k$; since $C^2 = n + 2k$ in the $\mathbb{F}_n$, by the triple point formula we must have $C^2 = -n - 2k$ in R. Therefore $R \cong \mathbb{F}_{n+2k}$. The original surface is still a minimal ruled surface, and its normal bundle is $-C$; hence it satisfies the contractibility criterion and can be contracted to a smooth curve. The resulting special fiber consists only of the surface R. This exhibits an explicit degeneration of $\mathbb{F}_n$ to $\mathbb{F}_{n+2k}$ for any $n \geq 0$ and any $k \geq 0$.

Example 1.9. Suppose that $\mathcal{X}$ is smooth, but that the special fiber X_0 is an irreducible surface with a single ordinary double point singularity at p. We will exhibit a degeneration to the resolution of X_0, after a base change. Blow up the double point p; the exceptional plane P will then meet the resolution S of the original fiber X_0 in a conic in P, which meets S in a (-2)-curve E of S. The new special fiber is not reduced, but is equal to $S+2P$. (The coefficient of 2 for P is because P was obtained by blowing up a point of multiplicity two.)

Make a base change of order two, by pulling back the family via a map $f:\Delta\to\Delta$ given by $f(t)=t^2$, and normalizing. This has the same effect as forming the double cover of the threefold branched at the special fiber $S+2P$; the normalization process is equivalent to removing all double components from this branch locus. Therefore the result is the same as forming the double cover branched over the surface S.

The new special fiber over the original Δ is $2S+2T$, where T is the double cover of P branched over the intersection with S, and where the coefficient for S doubles (because it is branched). Since the map f doubles the multiplicities, the fiber over the pullback is simply $S+T$.

Note that $T\cong\mathbb{F}_0$, and the double curve is now a curve of bidegree $(1,1)$; indeed, by the triple point formula, the double curve on S is still the (-2)-curve E, so the double curve must have self-intersection $+2$ on T, and this implies it has bidegree $(1,1)$. Therefore it is a section (in fact for either ruling) and by the contraction criterion T can be blown down to E (in two ways). The resulting degeneration has general fibers which are isomorphic to the general fibers of the original family, but the special fiber is now the resolution S of the surface X_0.

This is the first basic example of a *Brieskorn resolution*; see [B]. The two ways of blowing down T exhibits the simplest example of a *flip*.

2. Lecture Two

Why study degenerations? There are many reasons, but I think it is fair to say that two of them are preeminent. The first is to exploit semi-continuity: if one can prove a property about the degenerate object, which satisfies semicontinuity, then one obtains at least an inequality about the general object, and often this is enough to prove a precise statement. The other is to develop global formulas about a complete family, which depend on the degenerate fibers of the family.

Examples of the first type abound; we will see several later in the lectures. Examples of the second type are formulas for the topological invariants for example; or the Shioda-Tate formula for elliptic surfaces (which relates the rank of the Mordell-Weil group of sections to the types and number of singular fibers; see [M2] for an elementary account).

Here we make a small digression to an interesting set of examples which use a base change technique similar to that discussed at the end of the first lecture, and which relate to this second reason for studying degenerations. Any elliptic curve can be written in Weierstrass form

$$y^2=x^3+Ax+B$$

and the J-invariant of the curve is $J=4A^3/(4A^3+27B2)$. If A and B are constants, we have a single elliptic curve, with zero-point at infinity; however if $A=A(t,s)$ and $B=B(t,s)$ are homogeneous forms on $\mathbb{P}^1$, we have a family of elliptic curves over $\mathbb{P}^1$, with a section (at infinity). We must in fact have that A and B are forms of degree $4k$ and $6k$ respectively. If

$k = 1$ we have a rational elliptic surface; for $k = 2$ we have an elliptic $K3$ surface. Now the local Euler number and J-invariants of the possible singular fibers are:

type	Euler number	J
I_n	n	∞ for $n \geq 1$; arbitrary for $n = 0$
I_n^*	$n + 6$	∞ for $n \geq 1$; arbitrary for $n = 0$
II	2	0
III	3	1
IV	4	0
IV^*	8	0
III^*	9	1
II^*	10	0

Now for example a rational elliptic surface has total Euler number 12; this then limits the number and type of singular fiber. Twelve singular fibers of type I_1 (nodal rational curves) is one possibility, but there are many others. The possibilities have been totally classified by Persson [P] (see also [M1]).

It is an elementary exercise to classify all rational elliptic surfaces with section, having constant J and exactly two singular fibers; they are:

equation	singular fibers
$y^2 = x^3 + at^2s^2x + bt^3s^3$	I_0^*, I_0^*
$y^2 = x^3 + t$	II, II^*
$y^2 = x^3 + tx$	III, III^*
$y^2 = x^3 + t^2$	IV, IV^*

These are interesting surfaces in their own rights, but in fact they are useful for constructing hyperelliptic surfaces. Take one of the rational elliptic surfaces with constant J, having two singular fibers of type I_0^*, over 0 and ∞ say. Take the double cover of the base branched over 0, 1, ∞, and λ; this is a genus one curve. Pull back the family to this genus one curve, and normalize; the result is a hyperelliptic surface with $2K = 0$.

The same construction, made with higher-order covers of degree 3, 4, and 6 over the other three surfaces respectively, gives the primary families of hyperelliptic surfaces with $3K = 0$, $4K = 0$, and $6K = 0$.

As another example of singular fiber classification, one may consult [MP1], where all possible sets of singular fibers (all of type I_n) for $K3$ elliptic surfaces are classified.

It is fair to say that the classification of singular fibers for known families of general fibers is not very well fleshed out. We have seen that for curves, we have only trivial singular fibers if $X_t \cong \mathbb{P}^1$, and if X_t has genus one we have Kodaira's list. There is also a classification, much more complicated, for genus two families; but the main point is that any configuration of components in the special fiber for curves is possible, as long as the normal bundle formulas are satisfied (see [AW]).

For surfaces, the most successful classification result is for surfaces with $\kappa = 0$. This was accomplished in the early 1980s by Kulikov, Persson, and Pinkham (see [Ku], [PP]) (for $K3$ and abelian surfaces), with extension to Enriques and hyperelliptic surfaces by Morrison [Mo1].

The structure of the classification argument goes in a series of steps. To fix notation, we consider only degenerations of surfaces with $K = 0$, i.e., $K3$ or abelian surfaces. Let $\pi : \mathcal{X} \to \Delta$ be such a degeneration. Write the special fiber $X_0 = \sum_i n_i V_i$ as usual.

Since $K_{X_t} \equiv 0$, the canonical class of the threefold $\mathcal{X}$ must be concentrated on the special fiber, and we can write

$$K_{\mathcal{X}} = \sum_i m_i V_i$$

for integers m_i; note that since X_0 is lineary equivalent to zero, we have that the integers m_i are defined modulo the vector (n_i).

Step 1: Semistable Reduction This is an application of the general semistable reduction theorem, which says that after a base change and birational modifications, we may assume that the special fiber is reduced: $X_0 = \sum_i V_i$, i.e., $n_i = 1$ for every i. We also may assume that each V_i is smooth, and that the configuration has global normal crossings.

Step 2: Achieve $\mathbf{K = 0}$ This is a rather technical step in the proof; one proves that, after birational modifications of the central fiber, one can achieve $K_{\mathcal{X}} = 0$. The idea is that components V_i for which the K-multiplicity m_i is maximal should be contractible.

Let us illustrate why this should be true, with the much simpler situation of degenerations of elliptic curves. Assuming that the special fiber is reduced, choose a component V_0 of maximal K-multiplicity m_0; since these multiplicities are defined modulo $(1, 1, \ldots, 1)$, we may assume that $m_0 = 0$, and all $m_i \leq 0$. If we do not have $K = 0$ yet, we may assume that there is another component V_j such that $m_j < 0$, and that V_0 and V_j do intersect: $V_0 \cdot V_j > 0$. Then

$$\deg(K_{V_0}) = (K_{\mathcal{X}} + V_0) \cdot V_0 = V_0 \cdot V_0 + \sum_{i \neq 0} m_i V_i \cdot V_0.$$

Now $V_0^2 < 0$ as is the case with all components of a degeneration of curves; and the sum is negative, since each $m_i \leq 0$, and for $i = j$, we have $m_j < 0$ and $V_j \cdot V_0 > 0$. Therefore V_0 has a strictly negative canonical divisor, so that $V_0 \cong \mathbb{P}^1$. Hence $\deg(K_{V_0}) = -2$, so that we must have $V_0^2 = -1$, $m_j = 1$, $V_j \cdot V_0 = 1$, and $V_0 \cdot V_i = 0$ for any other i such that $m_i < 0$. In any case V_0 is a (-1)-curve in the special fiber, and we may contract V_0 and proceed by induction on the number of components.

This general approach for degenerations of surfaces runs into some trouble, because it shows that any component of maximum K-multiplicity in X_0 must be either a contractible $\mathbb{P}^2$ (which is good) or a ruled surface whose normal bundle meets the fiber with degree -1 (which is needed to contract the ruled surface to a curve). The problem is that these ruled surface may well not be minimal. Such a ruled surface is called *generically contractible.*

Kulikov's approach [Ku] is to use birational modifications to make such a component minimal; it is a delicate analysis. Persson and Pinkham (see [PP]) finesse the problem by introducing a category of *quasi-degenerations* in which the non-minimal fibers of generically contractible surfaces are removed from the threefold temporarily, with careful track being kept of how birational modifications of the rest of the threefold would affect the removed part. They prove that eventually the removed pieces, which are also undergoing birational modifications, are made minimal and can be contracted also.

A third approach, employing more recent techniques, would be to use Mori Theory.

Step 3: Classify The Components Once one has $X_0 = \sum_i V_i$ and $K_{\mathcal{X}} = 0$, we see that

$$K_{V_j} = (K_{\mathcal{X}} + V_j)|_{V_j} = V_j|_{V_j} \equiv \mathcal{O}_{V_j}(-D_j)$$

where $D_j = \sum_{i \neq j} C_{ji}$ is the divisor of the double curves on V_j. By the semistable reduction assumption, this divisor is a reduced curve with normal crossings on V_j. Surfaces with such an anti-canonical divisor are easily classified; in broad terms we have:

Proposition 2.1. *Suppose that V is a smooth surface with a reduced normal crossings divisor $D \in |K_V|$. Then V and D are on the following list:*

(a) *V is $K3$ or abelian, and $D = 0$*
(b) *V is a rational surface and D is a cycle of smooth rational curves*
(c) *V is a rational surface and D is a smooth curve of genus one*
(d) *V is an elliptic ruled surface and D is a smooth curve of genus one (which is a bisection for the ruling)*
(e) *V is an elliptic ruled surface and D consists of two disjoint genus one curves (both of which are sections for the ruling).*

Step 4: Determine The Configurations We must now take the above list of possible components, and put them together to make a special fiber X_0.

Proposition 2.2. *The possible semistable singular fibers of a degeneration $\pi : \mathcal{X} \to \Delta$ of $K3$ or abelian surfaces with $K_{\mathcal{X}} = 0$ are:*

(0) *A smooth $K3$ or abelian surface*
(I) *A cycle of elliptic ruled surfaces, meeting along disjoint sections; or a chain of surfaces meeting along genus one curves*
(II) *A union of rational surfaces meeting along anticanonical cycles of smooth rational curves.*

Indeed, since the configuration of surfaces are connected, if one finds a surface of type (a) (in Proposition (2.1), it must be the only surface, and we have a singular fiber of type (0).

If there is any component having a rational double curve, then by Proposition (2.1), all components must have rational double curves, and we have a singular fiber of type (II). We note that if we form the dual complex Γ (a vertex for each component, an edge for each double curve, and a triangle for each triple point), we must have that Γ is a triangulation of a closed 2-manifold without boundary.

Otherwise all components have only genus one double curves, and the only possibility is to have a singular fiber of type (I). We note in this case that if we have a chain of surfaces, at the ends of the chain we must have surfaces of types (c) or (d); and any interior surfaces in the chain (if the chain has length three or more) must be a surface of type (e). In the case of a cycle of surfaces, all the components must be of type (e).

Step 4: Topological Considerations

In order to distinguish between these possibilities, and to further refine the analysis, we invoke some topological and analytic considerations which are very useful in many situations. No doubt the most useful is the information one obtains from the Clemens-Schmidt exact sequence (see [C] or [Mo2] for details). One considers the map

$$\phi : \oplus_i H^1(V_i) \to \oplus_i H^1(C_{ij})$$

given in the obvious way by restriction. The dimension of the cokernel of ϕ must be even; let d be half this dimension. Then Clemens-Schmidt implies that

$$q(X_t) = h^1(\Gamma) + \sum_i q(V_i) - \sum_{i,j} g(C_{ij}) + d$$

and

$$p_g(X_t) = h^2(\Gamma) + \sum_i p_g(V_i) + d.$$

(In the formula for the irregularity q, each double curve C_{ij} is counted only once.)

Let us consider first degenerations of $K3$ surfaces. In this case $q = 0$ and $p_g = 1$.

Suppose that we have a degeneration of type (I), which is a chain of length k. Then Γ has trivial homology, and none of the components can have positive p_g, so that $d = 1$, by using the formula for $p_g(X_t)$. We also see that $\sum_i q(V_i) = k$, $k-1$, or $k-2$, depending on whether the ends of the chain are rational or elliptic ruled. All double curves are elliptic, so that $\sum g(C_{ij}) = k - 1$. Hence by the formula for $q(X_t)$, we see that

$$\sum_i q(V_i) = q(X_t) - h^1(\Gamma) + \sum g(C_{ij}) - d = 0 - 0 + (k-1) - 1 = k - 2$$

and therefore we conclude that the ends of the chain are rational.

If we have a degeneration of type (I) which is a cycle, then $h^1(\Gamma) = 1$ and $h^2(\Gamma) = 0$; all k components have $q(V_i) = 1$ and $p_g(V_i) = 0$ and all k double curves have genus one. Therefore the q formula gives $d = -1$, which is impossible; hence a $K3$ surface cannot degenerate in this way.

Suppose now that a degeneration of $K3$ surfaces is of type (II). In this case all surfaces are rational and all double curves are rational, so that the map ϕ is trivial, and $d = 0$; hence we have $h^1(\Gamma) = 0$ and $h^2(\Gamma) = 1$. The latter says that Γ is a triangulation of an orientable 2-manifold, and the former implies that it is a sphere.

One can analyze in a similar way the case of a degeneration of abelian surfaces, where $q = 2$ and $p_g = 1$. I leave it to the reader to check that if the degeneration is of type (I), it must be a cycle of elliptic ruled surfaces; and if the degeneration is of type (II), it must be a triangulation of the 2-torus $S^1 \times S^1$.

This proves the following more precise version of the previous Proposition:

Proposition 2.3. *The possible semistable singular fibers of a degeneration $\pi : \mathcal{X} \to \Delta$ of $K3$ surfaces with $K_{\mathcal{X}} = 0$ are:*

- (0) *A smooth $K3$ surface*
- (I) *A chain of $k \geq 2$ surfaces, meeting along smooth elliptic curves, with elliptic ruled surfaces as interior elements of the chain (meeting along disjoint sections), and rational surfaces at the two ends*
- (II) *A union of rational surfaces meeting along anticanonical cycles of smooth rational curves whose dual complex is a triangulation of the two-sphere.*

Proposition 2.4. *The possible semistable singular fibers of a degeneration $\pi : \mathcal{X} \to \Delta$ of abelian surfaces with $K_{\mathcal{X}} = 0$ are:*

- (0) *A smooth abelian surface*
- (I) *A cycle of elliptic ruled surfaces, meeting along smooth elliptic curves which are disjoint sections for the rulings*
- (II) *A union of rational surfaces meeting along anticanonical cycles of smooth rational curves whose dual complex is a triangulation of the two-torus.*

Step 5: Normal Forms

Given the previous analysis, one may ask if further birational modifications are possible in order to simplify in any way the possible singular fibers, and further restrict the list of degenerations.

It is not difficult to see that in the case of degenerations of $K3$ surfaces of type (I), one can achieve that all double curves (which have genus one) have self-intersection zero on both surfaces on which they lie. Indeed, for this, one only needs to "throw" (-1)-curves (which one finds on the rational surfaces at the end).

More interesting is the case of degenerations of type (II). In this case all double curves are rational, with exactly two triple points on each one; hence

$$C_{ij}^2 + C_{ji}^2 = -2$$

for each i and j. In fact one can arrange that they are both equal to -1:

Proposition 2.5. *(Minus One Theorem* [MM]*) Any type II degeneration of $K3$ surfaces can be modified, by a series of birational transformations, so that every double curve is of self-intersection -1 on both surfaces on which it lies.*

This greatly restricts the nature of the components in this case. First one notes that on each surface, the cycle of double curves must have length at most six: no cycle of (-1)-curves can exist of length seven or more. If we let s_j be the number of components with exactly j double curves, by the Euler number formula for triangulations of the sphere, we must have

$$\sum_j (6-j)s_j = 12$$

which also gives a strong restriction on the configuration.

These are the only combinatorial restrictions on the configuration however; any triangulation of the two-sphere occurs (see [F]).

3. Lecture Three

Let us turn our attention to the construction of degenerations. There are two basic techniques. The first is via deformation theory: one takes a proposed special fiber, and prove that it has a smooth deformation. The second is via direct construction, usually via some analysis of the varieties in question as divisors in an ambient variety. The simplest possible example is a conic in the plane degenerating to two lines. One step more complicated is a smooth quadric surface degenerating to two planes.

These last are examples of *embedded degenerations*, that is, degenerations that occur in projective space. To be precise, these are flat proper families $\pi : \mathcal{X} \to \Delta$ that comes together with a mapping $i : \mathcal{X} \to \mathbb{P}^r \times \Delta$ such that $\pi = i \circ p_2$, where $p_2 : \mathbb{P}^r \times \Delta \to \Delta$ is the second projection.

These are given by a line bundle $\mathcal{L}$ on $\mathcal{X}$, such that $h^0(X_t, \mathcal{L}|_{X_t}) = r$ for every t; the map of each fiber X_t into $\mathbb{P}^r$ (which need not be an embedding) is given by the sections of $\mathcal{L}|_{X_t}$. We usually focus only on the images of the fibers X_t in $\mathbb{P}^r$.

Example 3.1. The generalization of the conic degenerating to two lines in the plane is the case of a rational normal curve of degree m degenerating to a chain of m lines in $\mathbb{P}^m$. This is easily achieved by taking a product family of $\mathbb{P}^1$'s, and blowing up the central fiber $m-1$ times to create a chain of m smooth rational curves $V_1, \ldots, V_m$. Choose a point p_i in each of these curves, and a section D_i passing simply through p_i. The divisor $D = \sum_i D_i$ then determines a line bundle $\mathcal{L}$ which has degree m on the general fiber, and restricts to a degree one bundle on each component D_i of the special fiber. The relative sections of $\mathcal{L}$ embeds the general fiber as a rational normal curve of degree m in $\mathbb{P}^m$, and the special fiber as a chain of lines. Indeed, one may choose coordinates in the projective space so that if the coordinate points of $\mathbb{P}^m$ are denoted by $q_0, \dots, q_m$, then each D_i is the line joining q_{i-1} to q_i.

Example 3.2. We may make a similar construction by considering an I_m degeneration of elliptic curves, that is, a degeneration to a reduced cycle of smooth rational curves, of length m. Again taking a section meeting each component, and using the sum as giving the embedding line bundles, we see that the general fibers are mapped to elliptic normal curves of degree m in $\mathbb{P}^{m-1}$, and the special fiber is mapped to a cycle of lines that span this ambient space.

Example 3.3. We may take Example 3.1 and vary it by considering the divisor $\sum_i k_i D_i$ for arbitrary positive integers k_i. The sections of the embedding bundle then map the general fiber to a rational normal curve of degree $k = \sum_i k_i$ in $\mathbb{P}^k$, and the special fiber to a chain of rational normal curves of degree k_i.

The same variation can be made of course in the elliptic normal curve case, giving a degeneration to a cycle of rational normal curves.

Example 3.4. Consider again Example 3.1, and this time take the cross-product with $\mathbb{P}^1$, embedded as a line, and use the Segre embedding. The general fiber is a minimal ruled surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, embedded into $\mathbb{P}^{2m+1}$ by the bundle having bidegree $(1, m)$. (This is a rational normal surface scroll of degree $2m$.) The special fiber is a chain of $\mathbb{F}_0$'s of length m, meeting transversally along fibers, each embedded by the bundle having bidegree $(1, 1)$; i.e., each of these components is a smooth quadric.

Twists. Given an embedded degeneration (with the embedding defined by a line bundle $\mathcal{L}$) whose special fiber is $X_0 = \sum_i V_i$, one may obtain a radically different embedded degeneration (keeping the same embedding of the general fiber X_t) by employing a *twist* of the bundle $\mathcal{L}$.

Choose a component V_0 of the special fiber, and consider using, instead of the embedding bundle $\mathcal{L}$, the bundle

$$\tilde{\mathcal{L}} = \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-V_0).$$

We note that both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ restrict to the same bundle on the general fiber X_t for $t \neq 0$, and so we have not altered the embedding of the general fiber.

However for the special fiber, we note that

$$\tilde{\mathcal{L}}|_{V_i} = \mathcal{L}|_{V_i} \otimes \mathcal{O}_{V_i}(-C_{i0}) \text{ for } i \neq 0$$

and

$$\tilde{\mathcal{L}}|_{V_0} = \mathcal{L}|_{V_0} \otimes \mathcal{O}_{V_0}(\sum_{j \neq 0} C_{0j}).$$

In other words, we have subtracted the double curve C_{i0} from the embedding divisor for V_i $(i \neq 0)$ and we have added the divisor of the double curves for V_0.

We may employ this trick more generally, by twisting by any integer multiple of V_0. Indeed, we may twist by any divisor $D = \sum_i t_i V_i$.

Example 3.5. Consider Example 3.4, where we degenerated $\mathbb{F}_0$ embedded via the bundle of bidegree $(1,2)$ to two $\mathbb{F}_0$'s, each embedded via the bundle of bidegree $(1,1)$, (giving two quadrics in $\mathbb{P}^3$'s) meeting along a line which is a fiber of each $\mathbb{F}_0$. Take this example, and double the bundle: we obtain a degenration of $\mathbb{F}_0$ embedded into $\mathbb{P}^{14}$ via the bundle of bidegree $(2,4)$ as a surface of degree 16, to two $\mathbb{F}_0$'s, each embedded into a $\mathbb{P}^8$ via the bundle of bidegree $(2,2)$ as a surface of degree 8, meeting along a conic. This double curve is of type $(1,0)$ in each $\mathbb{F}_0$.

Now take this doubled example and twist it as above. One of the $\mathbb{F}_0$'s is now embedded by the bundle of bidegree $(3,2)$ and one by the bundle of bidegree $(1,2)$. The former is embedded as a surface of degree 12 into a $\mathbb{P}^{11}$, and the latter is embedded as a surface scroll of degree 4 in $\mathbb{P}^5$; they are still meeting along a conic.

In any case we have radically altered the degeneration by employing the twist.

Blowup and Twist. The twist technique becomes more interesting when we combine it with a blowup, as follows. Take an embedded degeneration of surfaces with embedding bundle $\mathcal{L}$ and special fiber $X_0 = \sum_i V_i$. Take either a point not on any double curve, or a smooth irreducible curve C in one of the surfaces, that meets the double curves transversally, and blow it up, obtaining an exceptional divisor V_0. Pull back the bundle $\mathcal{L}$ to the blowup, obtaining a bundle $\mathcal{M}$. If we simply use this pullback bundle to embed the fibers, it will certainly blow the exceptional divisor V_0 back down as part of the embedding mapping.

Instead, first twist, by a negative multiple of V_0, in order to make the restriction of the twisted bundle to V_0 sufficiently large to give a non-trivial map on V_0. In general, this will re-map all of the original surfaces in a different way (depending on how V_0 meets them), and will produce a new component in the limit, namely the image of V_0.

Let us consider several examples of this technique.

Example 3.6. Take the trivial family of $\mathbb{P}^2$'s, and use as the embedding bundle the bundle which is $\mathcal{O}(m)$ on each fiber. This gives a trivial family of m-Veronese surfaces. Choose a point in the special fiber and blow it up. One obtains as the new special fiber a union of two surfaces $\mathbb{P}$ and $\mathbb{F}$. The surface $\mathbb{P}$ is the exceptional divisor, which is a $\mathbb{P}^2$. The surface $\mathbb{F}$ is the blowup of the original central fiber, at one point; this is therefore isomorphic to the ruled surface $\mathbb{F}_1$. They meet along a smooth rational curve, which is the negative curve B on $\mathbb{F}_1$, and is a line on the $\mathbb{P}^2$.

Recall that the Picard group of $\mathbb{F}_1$ is generated by the negative curve B and the fiber F; the pullback H of a line in the plane is linearly equivalent to $B+F$, and is represented by a section of the ruling which is disjoint from B.

Pulling back the embedding bundle which is $\mathcal{O}(m)$ on the family of $\mathbb{P}^2$'s gives a bundle $\mathcal{M}$, which restricts to 0 on the surface $\mathbb{P}$, and to the bundle $\mathcal{O}(mH) = \mathcal{O}(mB+mF)$ on the surface $\mathbb{F}$. Of course it still restricts to $\mathcal{O}(m)$ on the general fiber.

Now execute the twist, by the bundle $\mathcal{O}_{\mathcal{X}}(-(m-1)\mathbb{P})$. This produces a new bundle $\tilde{\mathcal{M}}$; and we now have that

$$\tilde{M}|_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(m-1) \text{ and } \tilde{M}|_{\mathbb{F}} \cong \mathcal{O}_{\mathbb{F}}(B+mF)$$

while still restricting to $\mathcal{O}(m)$ on the general $\mathbb{P}^2$ fiber.

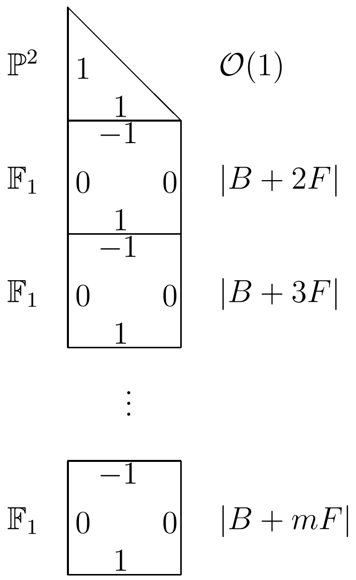
The bundle associated to the divisor $B+mF$ on $\mathbb{F}_1$ embeds $\mathbb{F}_1$ as a rational normal scroll of degree $2m-1$ in $\mathbb{P}^{2m}$. The curve B maps to a rational normal curve of degree $m-1$, and the curves in the system $|H|$ map to rational normal curves of degree m; the fibers are mapped to lines.

The bundle $\mathcal{O}_{\mathbb{P}}(m-1)$ of course maps the surface $\mathbb{P}$ as the $(m-1)$-Veronese.

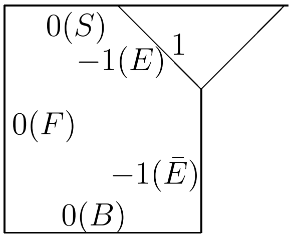
We have therefore constructed, via the blowup-and-twist technique, a degeneration of m-Veronese surfaces to a union of an $(m-1)$-Veronese surface and a rational normal scroll (isomorphic to $\mathbb{F}_1$) of degree $2m-1$, embedded via $|B+mF|$; they meet along a rational normal curve of of degree $m-1$, which is the negative section B of the scroll, and is the image of a line in the $(m-1)$-Veronese.

Example 3.7. We may take the above example, and iterate it, by taking the $(m-1)$-Veronese component, blowing up a point in it, and twisting again. The first time this will then produce three surfaces, the first $\mathbb{F}_1$ (still embedded by $|B+mF|$), the new $\mathbb{F}_1$ (this one embedded by $|B+(m-1)F|$) and the new $\mathbb{P}^2$ (embedded by $\mathcal{O}(m-2)$). They meet in a chain; the double curve in $\mathbb{P}^2$ is a line, and this $\mathbb{P}^2$ meets the new $\mathbb{F}_1$ along this line which is the curve B; the new $\mathbb{F}_1$ meets the first $\mathbb{F}_1$ along the double curve which is a curve in the system $|H|$ on the new $\mathbb{F}_1$ and the curve B on the first $\mathbb{F}_1$.

Continuing in this way $m-1$ times we find a degeneration of the m-Veronese which consists of m surfaces, meeting in a chain. At the 'top' of the chain we have $V_1 \cong \mathbb{P}^2$, embedded via $\mathcal{O}(1)$ as a plane. All other surfaces $V_2, \ldots, V_m$ are $\mathbb{F}_1$'s. If we arrange the indexing so that V_0 meets only V_1, and V_i meets only $V_{i\pm 1}$ for $2 \le i < m$, and V_m meets only V_{m-1}, we have that C_{01} is a line in V_0; $C_{i,i-1}$ is the negative curve B in each $V_i \cong \mathbb{F}_1$; and $C_{i,i+1}$ is a curve in the system H for each $2 \le i < m$. The component V_i is embedded via $B+iF$ as a rational normal scroll of degree $2i-1$:



Example 3.8. Let us consider now an interesting embedded degeneration of the rational normal scroll $\mathbb{F}_1$, embedded by the linear system $|B+kF|$ for some integer $k \ge 1$. Take the trivial family, with the constant bundle $\mathcal{O}(B+kF)$ on each fiber. Blow up a point in the special fiber which is not on the (-1)-curve B. One obtains a $\mathbb{P}^2$ as an exceptional surface; and the original $\mathbb{F}_1$ becomes a blowup of $\mathbb{F}_1$ at one point:



Here we have denoted by F the fiber class, B the (-1)-curve that was the negative section of the $\mathbb{F}_1$, S the section (formerly having self-intersection $+1$) through the blown-up point, E the exceptional curve for the blowup, and $\bar{E}$ the proper transform of the fiber of the $\mathbb{F}_1$ through the blown-up point. We have indicated the self-intersections in the above figure.

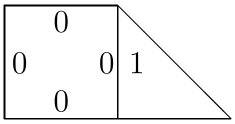
After pulling back the bundle to the blowup, the restriction to the general fiber is still given by $B + kF$, the restriction to the blowup of the $\mathbb{F}_1$ is $B + kF$, and the restriction to the exceptional $\mathbb{P}^2$ is trivial.

Now twist by $\mathcal{O}(-\mathbb{P}^2)$; this gives a new bundle $\mathcal{M}$, and now its restriction to the blown-up $\mathbb{F}_1$ is $B + kF - E$, while the restriction to the $\mathbb{P}^2$ is $\mathcal{O}(1)$.

Note that

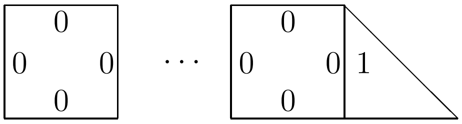
$$B + kF - E \cdot B = k - 1$$
$$B + kF - E \cdot F = 1$$
$$B + kF - E \cdot S = k - 1$$
$$B + kF - E \cdot E = 1$$
$$B + kF - E \cdot \bar{(}E) = 0$$

so that the linear system $B + kF - E$ contracts $\bar{E}$, and maps this surface to an $\mathbb{F}_0$, embedded by the linear system of bidgree $(1, k-1)$. The $\mathbb{P}^2$ is mapped by $\mathcal{O}(1)$ to a plane of course. This give an embedded degeneration of the scroll $\mathbb{F}_1$, embedded by $|B + kF|$, to a union of an $\mathbb{F}_0$ (embedded by the linear system of bidegree $(1, k-1)$) and a plane, meeting along a line (which is a member of the ruling of the $\mathbb{F}_0$):



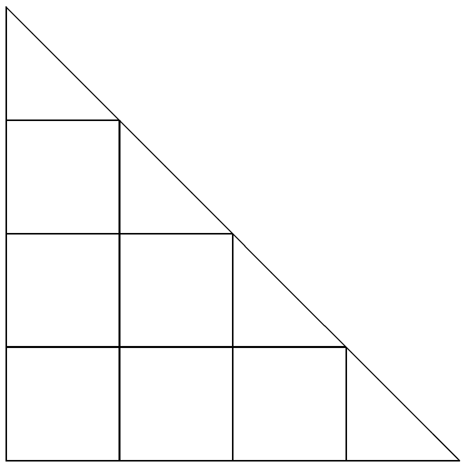
In the figure above, the horizontal curves indicated on the $\mathbb{F}_0$ are rational normal curves of degree $k - 1$; the vertical curves are lines.

Example 3.9. We may combine the previous example with Example 3.4, further degenerating the $\mathbb{F}_0$ to $k - 1$ quadrics; this then gives the degeneration:



whose general fiber is a scroll $\mathbb{F}_1$ embedded by $|B + kF|$.

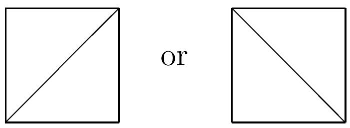
Example 3.10. At this point it is irresistable to combine Example 3.7 with Example 3.9. In Example 3.7, the m-Veronese was seen to degenerate to a plane together with $m - 1$ scrolls, each isomorphic to $\mathbb{F}_1$. Example 3.9 now provides a degeneration of each of the scrolls. The result is a configuration of m planes, together with $\binom{m}{2}$ quadrics; the configuration for $m = 4$ is depicted below:



All double curves in the configuration are lines. The squares of the figure are the quadrics, and the triangles are the planes. The double curves of the quadrics have self-intersection 0, while the double curves of the planes have self-intersection 1.

Note that the "vertices" of the figure can be taken to be the $\binom{m+2}{2}$ coordinate points of the ambient space for the embedding of the m-Veronese into the projective space of dimension $(m^2 + 3m)/2$.

Finally we may take this degeneration, and degenerate each quadric to two planes, keeping the double lines fixed. There are two ways to do this for each quadric:



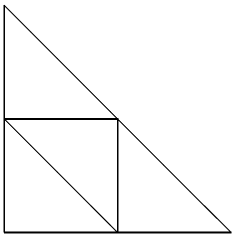
This provides a degeneration of the m-Veronese to m^2 planes; in fact we have $2^{\binom{m}{2}}$ ways to do this provided by this construction (although there are obviously symmetries).

Methods from toric geometry can also be used to make similar degenerations.

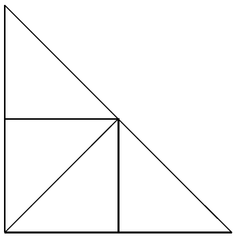
4. Lecture Four

In this last lecture let us connect up these constructions to the theme of secant varieties and defectivity that was prominent in the other lecture series here in Capri this week.

Consider first the two different degenerations of the 2-Veronese, provided by the construction in Example 3.10:

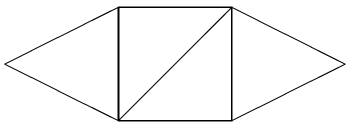


and



Note that the second one is a cone over a chain of 4 lines, and is therefore also a degeneration of the cone over a rational normal curve of degree 4.

These are not the only ways to take a union of four planes in $\mathbb{P}^5$ which meet along lines. For example, one might wonder if the configuration



is a possible degeneration of the 2-Veronese.

It is not. If it were, consider two points in the general fiber, degenerating to a point in each of the two planes at the ends. As is well-known, the 2-Veronese is defective, and the tangent planes to the 2-Veronese at any two points must meet. Passing to the limit, these tangent planes degenerate to planes that are tangent to this configuration, and must therefore degenerate to the two planes at the two ends. However these two planes do not meet. One cannot have a degenerating pair of planes that meet, which do not meet in the limit. Therefore the above configuration is not a limit of the 2-Veronese.

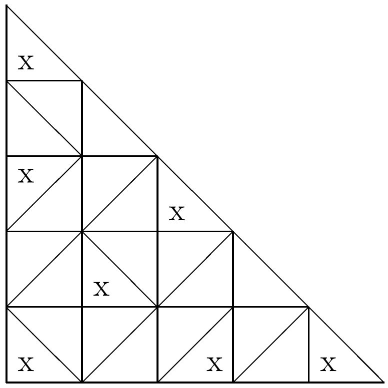
It is a degeneration of a rational normal scroll of degree 4 however; indeed, the image of $\mathbb{F}_0$ embedded via the linear system of bidegree $(1, 2)$ is such a scroll, and Example 3.4 shows that this degenerates to two quadrics meeting along a line. Further degenerating each quadric gives the above configuration of four planes as a limit.

A more systematic study of degenerations of surfaces to unions of planes was made by Zappa in a series of papers; see for example [Z]. Recently this has been revisited and extended by Calabri, Ciliberto, Flamini, and the author, see [CCFM1] and [CCFM2].

Let us close by proving that the linear system of quintic curves in $\mathbb{P}^2$ having seven general double points is empty. This is to be expected, since the dimension of the system of quintics is 20, and each of the seven double points imposes 3 conditions.

A quintic double at seven points $p_0, \dots, p_6$ would correspond to a hyperplane in $\mathbb{P}^{20}$ which is tangent to the 5-Veronese at the seven points. By Terracini's Lemma, such a hyperplane would correspond to a hyperplane that is tangent to the variety of 6-secants to the 5-Veronese. If we prove that no such quintic curve exists, and therefore prove that no such tangent hyperplane exists, we will have shown that this secant variety must be all of $\mathbb{P}^{20}$. This is the expected dimension of this secant variety, and so this would show that the 5-Veronese is not defective.

To prove that no such hyperplane exists, by semicontinuity, it is enough to show that no such hyperplane exists for a degeneration of the 5-Veronese. We will use a degeneration to 25 planes as in Example 3.10, and choose seven points (one in each of seven of the planes) and ask that a hyperplane be tangent to this configuration at these seven points. One configuration that works well for this purpose is the following:



The seven planes containing the seven limit points have been indicated by an 'x' in the figure above. Now a hyperplane which is tangent to a plane must in fact contain that plane. Therefore the hyperplanes tangent to this configuration at the seven limit points are in fact those hyperplanes which contain the seven indicated planes. However the reader can check that each of the 21 coordinate points of $\mathbb{P}^{20}$ (which can be taken to be the "vertices" of the configuration) are contained in exactly one of the seven planes. Therefore any hyperplane that contains the seven planes must contain these 21 points; but they span all of $\mathbb{P}^{20}$. This contradiction proves that no such hyperplane exists, and therefore the 5-Veronese is not defective.

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