Bielliptic Surfaces as Covers of Rational Surfaces

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Abstract

We present a construction of the bielliptic surfaces as covers of certain rational elliptic surfaces.

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Introduction

Bielliptic surfaces, sometimes referred to as hyperelliptic surfaces, form an important and interesting chapter in the story of the classification of surfaces, and have been classified for nearly a century. The standard construction of these surfaces are as quotients of products of elliptic curves. In this article we show that bielliptic surfaces can also be obtained naturally as covers of rather easily described rational surfaces. This affords both an opportunity to easily write equations for the bielliptic surfaces (at least birationally), and to readily construct their known degenerations.

In the next section we review the standard construction, primarily to set the stage and fix notation. In Section 2 we introduce the rational elliptic surfaces of which the bielliptic surfaces are covers. In Section 3 we describe the covering, which is obtained by a natural pullback procedure (followed by a normalisation). In the final section we indicate how the Type II degenerations of bielliptic surfaces arise quite naturally from this construction.

1 Bielliptic Surfaces

In the classification of complex surfaces, the coarsest grouping occurs by means of the κ invariant, which measures the growth rate of the space of sections of powers of the canonical bundle:

$$\dim(H^0(X, nK_X)) \text{ grows as } n^\kappa$$

where one uses the convention that $\kappa = -1$ (or sometimes $\kappa = -\infty$) if $\dim(H^0(X, nK_X)) = 0$ for all $n \geq 1$. For surfaces, κ is an integer at most two; the $\kappa = 2$ surfaces are those of general type, and the $\kappa = 1$ surfaces are elliptic. The pluricanonical maps for these types give a good start on their construction and more detailed understanding of their structure. For surfaces with $\kappa = 0$, other techniques must be brought into play. Their classification in broad terms is:

Theorem 1.1 *Let S be a smooth minimal complex algebraic surface with $\kappa_S = 0$. Then:*

(a) If $K_S = 0$ then S is either a $K3$ surface or an abelian surface. (These are distinguished by their irregularity: $q = 0$ for $K3$s and $q = 2$ for abelian surfaces.)
(b) If $K_S \neq 0$, then either $2K_S = 0$, $3K_S = 0$, $4K_S = 0$, or $6K_S = 0$. S is either an Enriques surface (having $q = 0$) or a bielliptic surface (having $q = 1$).

There is a well-known covering relationship between the two cases (a) and (b), which forms the heart of the classification of the Enriques and bielliptic surfaces:

Theorem 1.2 *Suppose S is a complex algebraic surface with $\kappa = 0$ but $K_S \neq 0$.*

(a) If S is Enriques then there exists a unique $K3$ surface W, an action of $\mathbb{Z}/2$ on W, acting without fixed points, such that $S \cong W/(\mathbb{Z}/2)$.
(b) If S is bielliptic then there exists an abelian surface A, which is a product of elliptic curves, and an action of a finite abelian group G on A, acting without fixed points, such that $S \cong A/G$.

For Enriques surfaces, there are also other alternate descriptions, for example, as a ramified double cover of a rational ruled surface. This was first given a modern exposure in Michael Artin's thesis [1], as well as in Boris Averbuch's thesis (for an exposition see [14]); one may also consult [8] and [5], although even the earliest constructions of Enriques surfaces were as double planes.

The precise analysis of the quotient construction for bielliptic surfaces goes back to Bagnera and De Franchis; this is described well in [3], [7], and [2]. We recapitulate here the basic details; for much more on the history, see [4].

Theorem 1.3 *Every bielliptic surface S is a quotient of a product $E \times F$ of elliptic curves by a finite abelian group $G = T \times C$, where T is a cyclic group of translations of F, C is a cyclic group of automorphisms of F fixing the elements of T, and G acts on E as a group of translations. The J-invariant of E is arbitrary; that of F may be constrained. The possibilities are given in the table below.*

$J(F)$	T	$\lvert C\rvert$	G	*Name*	K_S
arbitrary	0	2	$\mathbb{Z}/2$	$X_2(J)$	$2K_S = 0$
0	0	3	$\mathbb{Z}/3$	X_3	$3K_S = 0$
1	0	4	$\mathbb{Z}/4$	X_4	$4K_S = 0$
0	0	6	$\mathbb{Z}/6$	X_6	$6K_S = 0$
arbitrary	$\{0, \epsilon\}(2\epsilon = 0)$	2	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$Y_2(J)$	$2K_S = 0$
0	$\{0, (1+2\omega)/3, (2+\omega)/3\}$	3	$\mathbb{Z}/3 \times \mathbb{Z}/3$	Y_3	$3K_S = 0$
1	$\{0, (1+i)/2\}$	4	$\mathbb{Z}/2 \times \mathbb{Z}/4$	Y_4	$4K_S = 0$

The notation in the final three rows of the above table is that ϵ is a point of order two in F (in the fifth row), $F \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$ $(\omega^3 = 1)$ in the sixth row, and $F \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ $(i^2 = -1)$ in the last row.

There are two natural fibrations on S, given by the natural projections of the product; these are

$$\pi_F : S \to F/G \cong \mathbb{P}^1 \text{ and } \pi_E : S \to E/G.$$

E/G is an elliptic curve, and the fibration π_E is a fiber bundle with fibers isomorphic to F. The fibers of the fibration π_F are generically E; over a point of F/G that is the image of a point of F having isotropy group H, the fiber of π_F is isomorphic to E/H, with multiplicity $|H|$. These multiple fibers of the π_F fibration occur as multi-sections of the π_E fibration; an analysis of the actions give the following:

S	multiplicities of multiple fibers of π_F	degree over E/G via π_E
$X_2(J)$	$2,2,2,2$	$1,1,1,1$
X_3	$3,3,3$	$1,1,1$
X_4	$4,4,2$	$1,1,2$
X_6	$6,3,2$	$1,2,3$
$Y_2(J)$	$2,2,2,2$	$2,2,2,2$
Y_3	$3,3,3$	$3,3,3$
Y_4	$4,4,2$	$2,2,4$

The construction presented below is for the first four families, the X_i surfaces.

2 Rational Elliptic Surfaces

The rational elliptic surfaces of which the X_i bielliptic surfaces are covers have the following properties: they are rational elliptic surfaces over $\mathbb{P}^1$ having a section, they have constant J, and they have exactly two singular fibers. If one uses a Weierstrass equation to describe them, of the form

$$y^2 = x^3 + A(t)x + B(t),$$

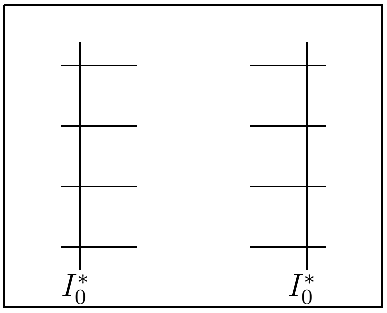
where t is an affine parameter on the base curve $\mathbb{P}^1$, then it is an elementary matter to classify these:

$A(t)$	$B(t)$	J	singular fibers	name	number of sections
at^2	bt^3	$4a^3/(4a^3+27b^2)$	I_0^*, I_0^*	$R_2(J)$	4
0	t^2	0	IV, IV^*	R_3	2
t	0	1	III, III^*	R_4	3
0	t	0	II, II^*	R_6	1

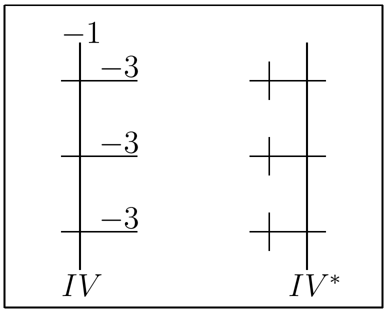
In the above table, the singular fibers occur over $t = 0$ and $t = \infty$. The indications of the type of singular fibers uses Kodaira's notation.

For each of the rational elliptic surfaces above, one can blow up the minimal elliptic model in order to achieve that the singular fibers have normal crossings.

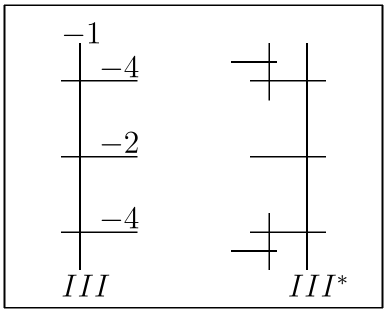
The surfaces $R_2(J)$, with two singular fibers of type I_0^*, already have normal crossings singular fibers, and so no blowups are necessary. For R_3, one blowup is required, at the concurrent point of the three components of the singular fiber of type IV. For R_4, two blowups are required, and for R_6, three blowups are required. The result in each case is depicted below. The self-intersections of all components having self-intersection different from -2 is indicated.



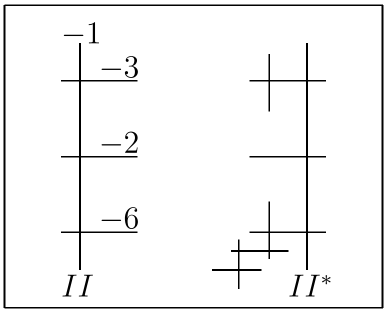
R_2



R_3



R_4



R_6

Each of these surfaces R_n $(n = 2, 3, 4, 6)$ has exactly two singular fibers, and each of the two singular fibers on R_n has a component having multiplicity n; this is the unique component that meets more than two others in each case.

For R_3, R_4, and R_6, one of the two singular fibers is minimal (the IV^*, III^*, and II^* fibers respectively) and one is not minimal, but has a (-1)-curve C_1 as a component. To fix notation, we have:

Surface	non-minimal singular fiber	canonical class
R_3	$C_{3a} + C_{3b} + C_{3c} + 3C_1$	$-F + C_1$
R_4	$C_{4a} + C_{4b} + 2C_2 + 4C_1$	$-F + C_2 + 2C_1$
R_6	$C_6 + 2C_3 + 3C_2 + 6C_1$	$-F + C_3 + 2C_2 + 4C_1$

We have indicated the components of the non-minimal singular fiber by using subscripts according to their self-intersection. The minimal elliptic models of each of the above rational elliptic surfaces have $K = -F$. This is therefore the canonical class of R_2. For the other ones, which are blowups of the minimal elliptic model, the canonical class is also indicated in the above table. We note in each case that $K = -F + (n-2)C_1$, modulo the other components; this formula also holds for R_2.

3 The Pullback

The bielliptic surfaces X_i will be covers of the rational elliptic surfaces R_i; in fact, the cover will be obtained by pulling back the elliptic fibration on R_i over $\mathbb{P}^1$ to an elliptic curve. This covering will be branched over $t = 0$, $t = \infty$, and two other points in general.

Lemma 3.1 *Fix $n \geq 2$, an elliptic curve D, and two points p_0 and p_∞ on D such that $p_0 - p_\infty$ has order n in $\mathrm{Pic}(D)$. Then there exists a map $f_n : D \to \mathbb{P}^1$ such that:*

(a) f_n has degree n, and sends p_0 to 0 and p_∞ to ∞;
(b) f_n has multiplicity n at both p_0 and p_∞;
(c) f_n either has exactly two other simple ramification points p_1 and p_2, or exactly one other double ramification point p_3 (which can only happen if $n \geq 3$);
(d) f_n has no other ramification points;
(e) if f_n has two other simple ramification points, then $p_1 + p_2 \equiv p_0 + p_\infty$, and if f_n has one other double ramification point, then $2p_3 \equiv p_0 + p_\infty$.

Proof: Since $p_0 - p_\infty$ has order n, there is a meromorphic function f_n on D such that $\mathrm{div}(f_n) = np_0 - np_\infty$; the associated holomorphic map to $\mathbb{P}^1$ satisfies (a) and (b) immediately. If R is the ramification divisor on D for the map, then $R = (n-1)p_0 + (n-1)p_\infty + R'$, where R' records the ramification apart from p_0 and p_∞. By the Hurwitz Formula, we have

$$0 = 2g(D) - 2 = -2n + \deg(R) = -2n + 2(n-1) + \deg(R') = -2 + \deg(R')$$

so that $\deg(R') = 2$. This forces $R' = p_1 + p_2$ or $R' = 2p_3$ as claimed, and proves (c) and (d). To prove (e), one uses the more refined Hurwitz formula: if t is the affine parameter on $\mathbb{P}^1$, and $\omega = dt/t$ is the meromorphic 1-form on $\mathbb{P}^1$ with simple poles at 0 and ∞, then

$$\mathrm{div}(f_n^*(\omega)) = f_n^*(\mathrm{div}(\omega)) + R \tag{3.2}$$

as divisors on D (see [11]). Now $\mathrm{div}(\omega)) = -0 - \infty$, so that $f_n^*(\mathrm{div}(\omega)) = -np_0 - np_\infty$; therefore the right-hand side of (3.2) is equal to $-p_0 - p_\infty + R'$.

The left-hand side is the divisor of a meromorphic 1-form on D, and since D is elliptic, this divisor is linearly equivalent to zero. This proves (e).

Q.E.D.

The construction of the bielliptic surfaces X_i are now obtained by pulling back the rational elliptic surfaces via the above map:

Theorem 3.3 *Fix $n = 2, 3, 4,$ or 6, and an elliptic curve D. Let $f_n : D \to \mathbb{P}^1$ be the elliptic curve covering $\mathbb{P}^1$ as in the above Lemma. Let $R_n \to \mathbb{P}^1$ be the rational elliptic surface described in the previous Section. Then the bielliptic surface X_n is the minimal model of the normalization of the pullback surface $D \times_{\mathbb{P}^1} R_n$. The natural map from X_n to D is the map π_E, the fiber bundle with fiber F.*

Proof: We will give the details for $n = 2$; the other three cases are quite similar and present no new ideas. In this case D is the double cover of $\mathbb{P}^1$ branched at 0, ∞, q_1, and q_2, where q_1 and q_2 are the images of the other two ramification points p_1 and p_2. Therefore the pullback is isomorphic to the double cover of R_n branched at the four fibers. Two of these fibers are smooth elliptic curves (the fibers over q_1 and q_2), and two are the singular fibers of type I_0^*. Taking the normalization of the pullback is equivalent to removing all the components of even multiplicity from the branch locus. This means removing the two double components F_0 and F_∞ from each of the two singular fibers. In this case the branch locus then consists of ten disjoint curves (the two smooth fibers, and the two sets of four disjoint (-2)-curves in the two singular fibers). Therefore the double cover is smooth. The canonical bundle of R_n is $-F$ (see [10]) and the line bundle of the branch locus is $4F - 2F_0 - 2F_\infty$. Half of this line bundle is $L = 2F - F_0 - F_\infty$, and so the canonical bundle of the double cover is the pullback of $K + L = F - F_0 - F_\infty$.

The pullback of the fiber class F of R_n is the class of two fibers F_1 and F_2 of the pullback (which are conjugate under the involution of the map $D \to \mathbb{P}^1$).

Let Q_1, Q_2, Q_3, Q_4 be the four other components of the singular fiber over 0. These lift to smooth rational curves Y_1, Y_2, Y_3, Y_4 on the pullback, which are (-1)-curves on the pullback. The inverse image of F_0 is an elliptic curve G_0, and the fiber over the point of D lying over 0 is exactly $G_0 + Y_1 + Y_2 + Y_3 + Y_4$. After blowing down the Y_i's we have that G_0 is the fiber. The same is true over the fiber over ∞: the inverse image G_∞ of F_∞ is the fiber (after blowing down the other four (-1)-curves).

Therefore, the canonical bundle of the pullback, after the eight (-1)-curves are blown down, is equal to $F_1 + F_2 - G_0 - G_\infty$. Since $F_1 + F_2$ consists of two fibers conjugate under the involution, as a divisor it is linearly equivalent to $2G_0$. Therefore the canonical bundle is linearly equivalent to $G_0 - G_\infty$.

This is clearly a divisor of order two; hence this minimalization of the normalization of the pullback is indeed a surface with $2K = 0$ but $K \neq 0$; since it has a map to an elliptic curve, it must have $q \geq 1$. Therefore it is a bielliptic surface, and must be either X_2 or Y_2. These are distinguished by the existence of four sections.

Q.E.D.

The above detail for the $n = 2$ case can also be carried out for the other three cases. However the reader may find it more comfortable to execute the necessary constructions in a slightly different order. If we take each R_n and contract all components of the two singular fibers except the component of multiplicity n, we obtain a rational elliptic surface which is not smooth, but has two singular fibers which are smooth rational curves of multiplicity n. Call these models U_n. Each U_n has canonical class given by $K = -F + (n-2)C_1$, where C_1 is the smooth rational curve fiber which comes from the non-minimal singular fiber.

If $f : X_n \to U_n$ is the covering (obtained by pulling back via f_n and normalizing), the ramification of f occurs over two fibers only (the fibers S_1 over p_1 and S_2 over p_2), or doubly over a single fiber (the fiber S_3 over p_3 in that case). A local analysis shows that X_n is smooth. A canonical divisor on X_n is obtained by pulling back a canonical divisor from U_n and adding the ramification divisor. Therefore the divisor

$$f^*(-F + (n-2)C_1) + S_1 + S_2 \text{ or } f^*(-F + (n-2)C_1) + 2S_3$$

is a canonical divisor on X_n.

Now $f^*(C_1)$ is the fiber S_0 over $0 \in D$; moreoever, $f^*(F) = nS_0$, by taking the fiber over $0 \in \mathbb{P}^1$. Therefore

$$S_1 + S_2 - 2S_0 \text{ or } 2S_3 - 2S_0$$

is a canonical divisor on X_n. By Lemma 3.1(e), if S_∞ is the fiber of X_n over p_∞, we see that

$$S_0 + S_\infty - 2S_0 = S_\infty - S_0$$

is a canonical divisor on X_n. This has order n.

4 Degenerations

These constructions of the bielliptic surfaces are especially conducive to understanding and constructing their Type II degenerations. These were classified by D. Morrison in [12], using techniques of Victor Kulikov [9], and following the approach of Persson and Pinkham [13]; see also [6].

Again, this is most easily illustrated in the $n = 2$ case. In this case as the two other ramification points p_1 and p_2 on D approach p_0 and p_∞, D degenerates to a union of two rational curves, meeting transversally at two points. The bielliptic surface then degenerates to two copies of R_n, and the resolution of the singularities requires eight $\mathbb{P}^2$s in order to have a smooth threefold family. This is the Type II degeneration described precisely in [12].

A simpler version of the above occurs when the two points p_0 and p_∞ come together. In this case D degenerates to a nodal rational curve, and the bielliptic surface degenerates to a non-normal surface which is R_2 glued to itself along the two singular fibers. In this case the resolution of singularities only requires four $\mathbb{P}^2$s to make the threefold family smooth.

The above degeneration can be performed with the other three families also; the result is a degeneration of the bielliptic surface to R_n $(n = 3, 4, 6)$ glued to itself along the two singular fibers. The resolutions required to make the threefold smooth are precisely those described in [12]. We leave the details to the reader.

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It is a great pleasure to dedicate this small contribution to Professor Michael Artin, who first introduced me to elliptic surfaces, taught me how important they were, and enthusiastically communicated his pleasure in studying them; it is especially this last aspect that I have come to enjoy and appreciate more and more deeply as the years have gone by.

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