Quadratic finite-volume methods for elliptic and parabolic problems on quadrilateral meshes: optimal-order errors based on Barlow points

MIN YANG*
Department of Mathematics, Yantai University, Yantai, Shandong, China
*Corresponding author: yang@ytu.edu.cn

JIANGGUO LIU
Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874, USA
liu@math.colostate.edu

AND

YANPING LIN
Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong and Department of Mathematical and Statistics Science, University of Alberta, Edmonton, AB, Canada
T6G 2G1
malin@polyu.edu.hk yanlin@ualberta.ca

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This paper presents quadratic finite-volume methods for elliptic and parabolic problems on quadrilateral meshes that use Barlow points (optimal stress points) for dual partitions. Introducing Barlow points into the finite-volume formulations results in better approximation properties at the cost of loss of symmetry. The novel ‘symmetrization’ technique adopted in this paper allows us to derive optimal-order error estimates in the $H^1$- and $L^2$-norms for elliptic problems and in the $L^\infty(H^1)$- and $L^\infty(L^2)$-norms for parabolic problems. Superconvergence of the difference between the gradients of the finite-volume solution and the interpolant can also be derived. Numerical results confirm the proved error estimates.

Keywords: Barlow points; error estimation; elliptic boundary value problems; finite-volume element methods; parabolic equations; quadrilateral meshes.

1. Introduction

Finite-volume methods have been widely used in scientific computing and engineering applications due to their local conservation properties and easy implementation. However, most finite-volume methods are low order. Development of higher order finite-volume methods has been an active research area, as reflected in Cai et al. (2003), Chen (2010), Chen et al. (2011), Gao & Wang (2010), Hackbusch (1989), Hyman et al. (1992), Plexousakis & Zouraris (2004), Wang & Gu (2010), Xu & Zou (2009), Yang (2006), Yang et al. (2009), Yang & Liu (2011), Yu & Li (2010, 2011) and the references therein.

This paper is a continuation of our efforts in Yang (2006), Yang et al. (2009), Yang & Liu (2011) on developing quadratic finite-volume methods for elliptic and parabolic problems. It was discussed in Yang & Liu (2011) that the quadratic finite-volume method based on the Simpson quadrature for parabolic problems on quadrilateral meshes has an optimal convergence rate in the $L^2(H^1)$-norm, but there are technical difficulties in deriving error estimates in the $L^\infty(H^1)$- and $L^\infty(L^2)$-norms that are
similar to those for the finite-element methods. The main reason for this deficiency is that we were unable to derive an optimal $L^2$-norm error estimate for the Ritz projection. Such deficiency exists on triangular and quadrilateral meshes, see Li et al. (2000), Liebau (1996), Xu & Zou (2009) and Yang (2006). In this paper, we utilize Barlow points (Barlow, 1976) for constructing dual volumes and developing quadratic finite-volume methods for elliptic and parabolic problems on quadrilateral meshes. Now we are able to derive optimal-order error estimates in the $L^\infty(H^1)$- and $L^\infty(L^2)$-norms. We have also superconvergence for the errors between the interpolation and the finite-volume solution. A novel contribution of this paper is the ‘symmetrization’ technique that enables us to overcome certain difficulties in the theoretical analysis.

Throughout the paper, we use the standard notations for the Sobolev spaces $W^{m,p}(\Omega)$ with the norm $\| \cdot \|_{m,p,\Omega}$ and the seminorm $| \cdot |_{m,p,\Omega}$. We also denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and skip the index $p = 2$ and the domain $\Omega$, when there is no ambiguity, that is, $\| u \|_{m,p,\Omega} = \| u \|_{m,\Omega}, \| u \|_m = \| u \|_{m,\Omega}$. The same convention is adopted for the seminorms. We will also use $A \lesssim B$ and $B \gtrsim A$ to denote $A \leq CB$, where $C$ is an absolute constant that may take different values in different appearances but is independent of spatial and temporal discretizations. Moreover, $A \sim B$ denotes that $A \lesssim B$ and $B \lesssim A$.

The rest of this paper is organized as follows. In Section 2, we discuss Barlow points, construction of dual volumes and mesh assumptions. Finite-volume methods for elliptic and parabolic problems on quadrilateral meshes are presented in Section 3. Section 4 is devoted to error estimation of the developed finite-volume schemes. Section 5 presents numerical results to illustrate the error estimates. Section 6 concludes the paper with some remarks.

2. Quadratic finite volumes and mesh assumptions

Let $\Omega_h = \{ Q \}$ be a quadrilateral partition of $\Omega$, where any two closed quadrilaterals share a common edge, vertex or nothing. Let $\hat{Q} = [-1, 1]^2$ be the reference element in the $\hat{x}\hat{y}$-plane. For each element $Q \in \Omega_h$, there exists a bijective bilinear mapping $F_Q: \hat{Q} \rightarrow Q$ satisfying

$$ F_Q(\hat{P}_i) = P_i, \quad 1 \leq i \leq 4. $$

Let $J_{F_Q}$ be the Jacobian of $F_Q$ at $\hat{x}$ and $J_{F_Q} = \det J_{F_Q}$, and accordingly, $J_{F_Q^{-1}}$ the Jacobian matrix of $F_Q^{-1}$ at $x$ and $J_{F_Q^{-1}} = \det J_{F_Q^{-1}}$. Based on the partition $\Omega_h$, we define $S_h$ as the standard conforming finite-element space of piecewise affine biquadratic functions

$$ S_h = \{ v \in H^1_0(\Omega): v|_Q = \hat{v} \circ F_Q^{-1}, \hat{v}|_{\hat{Q}} \text{ is biquadratic, } \forall Q \in \Omega_h; v|_{\partial\Omega} = 0 \}. \tag{2.1} $$

Let $I_h: H^1_0(\Omega) \cap H^3(\Omega) \rightarrow S_h$ be the usual nodal interpolation operator. It is well known (see Brenner & Scott, 2008) that

$$ \| u - I_h u \|_r \lesssim h^{3-r}\| u \|_3, \quad 0 \leq r \leq 2. \tag{2.2} $$

In order to establish finite-volume element schemes, we introduce a dual partition $\Omega^*_h$, whose elements are called control volumes. As shown in Fig. 1, each edge of $Q \in \Omega_h$ is partitioned into three segments so that the ratio of these segments is $1 : \sqrt{3} + 1 : 1$. We connect these partition points with line segments to the corresponding ones on the opposite edge. This way, each quadrilateral in $\Omega_h$ is divided into nine sub-quadrilaterals $Q_z, z \in Z_h(Q)$, where $Z_h(Q)$ is the set of the vertices, the midpoints of edges and the centre of $Q$. For each node $z \in Z_h = \bigcup_{Q \in \Omega_h} Z_h(Q)$, we associate a control volume $V_z$, which is the union of the subregions $Q_z$ containing the node $z$. Therefore, we obtain a collection of
control volumes covering the domain $\Omega$. This is the dual partition $\Omega_h^*$ of the primal partition $\Omega_h$. We denote the set of interior nodes of $Z_h$ by $Z_0^h$.

**Remark 2.1** The dual partition introduced here is different from the one defined in Yang & Liu (2011), where the sub-quadrilaterals $Q_z$ are built by using the points related to the Simpson quadrature and the partition ratio is 1:4:1. The control volumes in this paper are based on the Barlow points (the optimal stress points). The set of the Barlow points for one-dimensional Lagrange quadratic finite element is $N_2 = F \hat{N}_2$, where $F$ is an invertible affine mapping from the reference element $[-1, 1]$ to a generic interval $[a, b]$, and

$$\hat{N}_2 = \left\{ -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\}.$$

Let $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and $L_i(x)$, $i = 1, 2, 3$ be the corresponding Lagrange quadratic basis functions. The derivative of the interpolant at the Barlow points satisfies (Barlow, 1976)

$$\left. \left( x^3 - \sum_{i=1}^{3} x_i^3 L_i(x) \right) \right|_{x = \pm \sqrt{3}/3} = 0,$$

which means that the stresses are the most accurate at the Barlow points. The property (2.3) is critical for obtaining a superconvergence result for the term $a_h(u - I_hu, I_h^* v_h)$; see Lemma 4.3.

**Remark 2.2** Finite-volume methods based on the Barlow points (the optimal stress points) have been previously studied on one-dimensional (see, e.g., Gao & Wang, 2010) and two-dimensional rectangular meshes, see, e.g., Wang & Gu (2010) and Yu & Li (2010, 2011). But the extension of the methods to
more general quadrilateral meshes is not straightforward. The distortion of transformation might destroy
the superconvergence property. A delicate analysis should be established to measure the effect of such
distortion.

We make some assumptions on the quadrilateral mesh $\Omega_h$ as follows. For any $Q \in \Omega_h$, let $h_Q$ be its
diameter, $h'_Q$ the smallest length of the edges and $\theta_Q$ any interior angle. We set $h = \max_{Q \in \Omega_h} h_Q$.

(1) **Mesh Assumption A.** The mesh $\Omega_h = \{Q\}$ is regular, that is, there exist two positive constants $\sigma$
and $\gamma$ such that

$$h_Q/h'_Q \leq \sigma, \quad |\cos \theta_Q| \leq \gamma < 1, \quad \forall Q \in \Omega_h. \quad (2.4)$$

(2) **Mesh Assumption B.** The quadrilateral mesh is ‘asymptotically parallelogram’. Namely, for each
element $Q \in \Omega_h$, one has

$$\|P_1 - P_2 + P_3 - P_4\| = O(h^2), \quad (2.5)$$

where $P_i, i = 1, 2, 3, 4$ are the four vertices of $Q$.

(3) **Mesh Assumption C.** Any two adjacent quadrilaterals (see Fig. 2) form an $h^2$-parallelogram in
the sense that

$$\|P_4 + P'_1 - 2P_3\| = O(h^2). \quad (2.6)$$

These assumptions have been respectively adopted in Arnold et al. (2002), Chou & He (2002), Ewing
et al. (1999) and Yang & Liu (2011).

### 3. Finite-volume schemes for elliptic and parabolic problems

We first consider a model elliptic boundary value problem

$$\begin{cases}
\nabla \cdot (-a(x)\nabla u) = f(x), & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases} \quad (3.1)$$

where $\Omega$ is a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial \Omega$ and $x = (x, y)$. It is assumed that
$f(x) \in L^2(\Omega)$ and $a(x)$ is Lipschitz continuous and bounded almost everywhere with positive lower and
upper bounds $a_*$ and $a^*$. 
Given a node \( z \in Z_h^0 \), integrating the first equation in (3.1) over the control volume \( V_z \) and applying the Green’s formula, we obtain

\[
- \int_{\partial V_z} a \nabla u \cdot n \, ds = \int_{V_z} f \, dx,
\]

(3.2)

where \( n \) denotes the unit outer normal on \( \partial V_z \).

We further introduce a transfer operator \( I_h^*: S_h \rightarrow S_h^* \) from the trial space to the test space such that

\[
I_h^* v = \sum_{z \in Z_h^0} v(z) \Psi_z,
\]

(3.3)

where

\[
S_h^* = \{ v \in L^2(\Omega) : v|_{V_z} \text{ is constant}, \forall z \in Z_h^0, v|_{V_z} = 0, \forall z \in \partial \Omega \},
\]

(3.4)

and \( \Psi_z \) is the characteristic function of the control volume \( V_z \). Then we multiply (3.2) by \( v_h(z) \) and sum over all \( z \in Z_h^0 \) to obtain an integral conservation form for elliptic problem

\[
a_h(u, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in S_h,
\]

(3.5)

where the bilinear form \( a_h(\cdot, \cdot) \) is defined as: for any \( u \in H^1_0(\Omega), v_h \in S_h, \)

\[
a_h(u, I_h^* v_h) = - \sum_{z \in Z_h^0} v_h(z) \int_{\partial V_z} a \nabla u \cdot n \, ds.
\]

(3.6)

A finite-volume scheme for elliptic problem (3.1) is formulated as: Seek \( u_h \in S_h \) such that

\[
a_h(u_h, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in S_h.
\]

(3.7)

A model parabolic initial boundary value problem can be formulated as

\[
\begin{cases}
    u_t - \nabla \cdot (a(x) \nabla u) = f(x, t), & (x, t) \in \Omega \times (0, T], \\
    u = 0, & (x, t) \in \partial \Omega \times (0, T], \\
    u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]

(3.8)

where \( f(x, t) \in L^2(\Omega) \) for \( t \in [0, T] \) and other terms follow the same assumptions for the elliptic problem.

A semidiscrete finite-volume scheme for (3.8) is accordingly defined as: Seek \( u_h(t) \in S_h \) for \( t \in (0, T] \) such that for any \( v_h \in S_h, \)

\[
(u_{h,t}, I_h^* v_h) + a_h(u_h, I_h^* v_h) = (f, I_h^* v_h)
\]

(3.9)

with an initial approximation \( u_h(0) \) given by \( u_h(0) = R_h u_0 \), where \( R_h : H^1_0(\Omega) \cap H^3(\Omega) \rightarrow S_h \) is the elliptic (Ritz) projection defined by

\[
a_h(R_h u, I_h^* v_h) = a_h(u, I_h^* v_h), \quad \forall v_h \in S_h.
\]

(3.10)
Remark 3.1 Let \( \Phi_z : z \in \mathbb{Z}_n \) and \( \Psi_z : z \in \mathbb{Z}_n \) be the standard basis functions of \( S_h \) and \( S_h^n \), respectively. Then scheme (3.9) can be rewritten as a system of ordinary differential equations

\[
\mathcal{M} \alpha'(t) + \mathcal{S} \alpha(t) = \mathcal{f}(t), \quad 0 \leq t \leq T; \quad \alpha(0) = \beta,
\]

where \( \mathcal{M} = (\Phi_z, \Psi_w)_{z,w} \) and \( \mathcal{S} = (a_h(\Phi_z, \Psi_w))_{z,w} \) are the mass and stiffness matrices, respectively, and \( \alpha(t) \) and \( \beta \) are vectors of the nodal values of \( u_h(t) \) and \( \mathcal{R}_hu_0 \), respectively. From Lemmas 4.1 and 4.4, we know that both \( \mathcal{M} \) and \( \mathcal{S} \) are invertible. This implies that there exists a unique solution \( u_h(\cdot, t) \) on \( \Omega \times [0, T] \).

Let \( N \) be a positive integer. For simplicity of presentation, we consider a uniform time step \( \Delta t = T/N \) and set \( t_n = n \Delta t \). Let \( t_i, t_j \) be the midpoints of the edges of \( \mathcal{Q} \) and denote its vertices by \( x_{ij}, x_{i+1,j}, x_{i+1,j+1}, x_{i,j+1} \), corresponding to \( P_1, P_2, P_3, P_4 \) in Fig. 2. Let \( v_i, v_j = 0 \) or \( \frac{1}{2} \). Then the midpoints of the edges of \( \mathcal{Q} \) are denoted by \( x_{i+v_i,j+v_j} \), where \( v_i + v_j = \frac{1}{2} \), and the centre of \( \mathcal{Q} \) denoted by \( x_{i+1/2,j+1/2} \).

A Crank–Nicolson fully discrete (Thomée, 2006) finite-volume scheme for (3.8) seeks \( u_h^n = u_h(n\Delta t) \) such that for any \( v_h \in S_h \),

\[
(\tilde{\mathcal{M}}u_h^n + I^h v_h) + a_h(u_h^{n,1/2}, I^h v_h) = (f^{n,1/2}, I^h v_h), \quad n \geq 1,
\]

with an initial approximation given by \( u_h^0 = \mathcal{R}_hu_0 \).

Remark 3.2 The scheme (3.12) takes a form as follows

\[
(\mathcal{M} + \frac{1}{2} \Delta t \mathcal{S}) \alpha^n = (\mathcal{M} - \frac{1}{2} \Delta t \mathcal{S}) \alpha^{n-1} + \Delta t \mathcal{f}^{n,1/2},
\]

Generally, \( \mathcal{M} \) and \( \mathcal{S} \) are not symmetric. But by Lemmas 4.1 and 4.4, we know that both \( \mathcal{M} + \mathcal{M}^T \) and \( \mathcal{S} + \mathcal{S}^T \) are positive definite. So, for any nonzero vector \( \mathbf{x} \),

\[
\mathbf{x}^T(\mathcal{M} + \frac{1}{2} \Delta t \mathcal{S})\mathbf{x} = \frac{1}{2} \mathbf{x}^T(\mathcal{M} + \mathcal{M}^T)\mathbf{x} + \frac{1}{4} \Delta t \mathbf{x}^T(\mathcal{S} + \mathcal{S}^T)\mathbf{x} > 0,
\]

which means \( \mathcal{M} + \frac{1}{2} \Delta t \mathcal{S} \) is invertible. Therefore, (3.12) can be solved uniquely at each time step.

4. Error estimates

For simplicity of presentation, we assume that the coefficient \( a(x) \equiv 1 \) in this section. If \( a(x) \) is non-constant, we can have a perturbation argument by taking the piecewise constant approximation in each element \( \mathcal{Q} \). Then all results will still hold for \( h \) small enough; see Yang & Liu (2011) for details.

4.1 Basic approximation properties

It is assumed that there exist a pair of integers \( n_x, n_y \) such that the cardinality of \( \Omega_h \) is equal to \( n_x n_y \), and we can assign each \( \mathcal{Q} \in \Omega_h \) a pair of integers \((i,j)\), where \( 0 \leq i \leq n_x - 1, \ 0 \leq j \leq n_y - 1 \). Thus we label \( \mathcal{Q} \) by subscripts \((i,j)\) and denote its vertices by \( x_{i,j}, x_{i+1,j}, x_{i+1,j+1}, x_{i,j+1} \), corresponding to \( P_1, P_2, P_3, P_4 \) in Fig. 2. Let \( v_i, v_j = 0 \) or \( \frac{1}{2} \). Then the midpoints of the edges of \( \mathcal{Q} \) are denoted by \( x_{i+v_i,j+v_j} \), where \( v_i + v_j = \frac{1}{2} \), and the centre of \( \mathcal{Q} \) denoted by \( x_{i+1/2,j+1/2} \).
Similar to Yang & Liu (2011), we now define some discrete norms on $S_h$. For any $u_h \in S_h$,

$$||u_h||^2_0 = (u_h, I_h^* u_h),$$  \hspace{1cm} (4.1)

$$|u_h|^2 = \sum_{Q \in \mathcal{Q}_h} |u_h|^2_{1,h,Q} = \sum_{Q \in \mathcal{Q}_h} \left[ \sum_{\nu_i = 1/2, 1 \leq \nu_j = 0, 1/2, 1} (\delta_i u_h(x_{i+\nu_i, j+\nu_j}))^2 + \sum_{\nu_i = 0, 1/2 \leq \nu_j = 1/2, 1} (\delta_j u_h(x_{i+\nu_i, j+\nu_j}))^2 \right] ,$$ \hspace{1cm} (4.2)

where

$$\delta_i u_h(x_{i+\nu_i, j+\nu_j}) = u_h(x_{i+\nu_i, j+\nu_j}) - u_h(x_{i+\nu_i, j+\nu_j-1/2}),$$

$$\delta_j u_h(x_{i+\nu_i, j+\nu_j}) = u_h(x_{i+\nu_i, j+\nu_j}) - u_h(x_{i+\nu_i, j+\nu_j-1}).$$

The following lemma reveals that the discrete norms are equivalent to the continuous norms.

**Lemma 4.1** Assume that $\mathcal{Q}_h$ satisfies Mesh Assumption A. For any $u_h \in S_h$, we have

$$||u_h||_0 \sim ||u_h||_0,$$ \hspace{1cm} (4.3)

$$|u_h|_{1,h} \sim |u_h|_1.$$ \hspace{1cm} (4.4)

**Proof.** Since the mesh is regular, we have

$$h_Q \tilde{u}_h \leq ||J_{F_0}^{1/2} u_h ||_{0,Q} \leq ||J_{F_0}^{1/2} \tilde{u}_h ||_{0,Q},$$

where

$$||J_{F_0}^{1/2} ||_{\infty,Q} \leq h_Q^{-2}, \hspace{1cm} ||J_{F_0} ||_{\infty,Q} \leq h_Q^2.$$ 

Therefore,

$$h_Q \tilde{u}_h \leq ||u_h ||_{0,Q} \leq h_Q \tilde{u}_h.$$ \hspace{1cm} (4.5)

For $\int_Q u_h I_h^* u_h \, dx$, we have a similar estimate as follows

$$h_Q^2 \int_Q \tilde{u}_h^2 I_h \tilde{u}_h \, d\hat{x} \leq h_Q^2 \int_Q u_h I_h^* u_h \, dx \leq h_Q^2 \int_Q \tilde{u}_h^2 I_h \tilde{u}_h \, d\hat{x}.$$ \hspace{1cm} (4.6)

Let

$$\varphi_1(x) = \frac{1}{2} x(x-1), \hspace{1cm} \varphi_2(x) = (1-x)(x+1), \hspace{1cm} \varphi_3(x) = \frac{1}{2} x(x+1)$$

be the local quadratic basis on $[-1, 1]$. Let $\psi_i(x)$, $1 \leq i \leq 3$ be the characteristic functions associated with the partitions $[-1, -\sqrt{3}/3], [-\sqrt{3}/3, \sqrt{3}/3]$ and $[\sqrt{3}/3, 1]$, respectively. Then using the standard tensor-product basis and the resulting interpolation form of $\tilde{u}_h$ on the reference element, we obtain
immediately
\[
\| \hat{u}_h \|_{0, Q}^2 = u_Q(\mathcal{G}_1 \otimes \mathcal{G}_1)u_Q^T, \quad \int_Q \hat{u}_h \hat{u}_h \hat{I}^* h \hat{u}_h \hat{x} = u_Q(\mathcal{G}_2 \otimes \mathcal{G}_2)u_Q^T = \frac{1}{2} u_Q(\mathcal{G}_2 \otimes \mathcal{G}_2 + \mathcal{G}_2^T \otimes \mathcal{G}_2^T)u_Q^T, \]

where \( u_Q \in \mathbb{R}^9 \) is a vector consisting of the nodal values of \( u_h \) on \( Q \) and

\[
\mathcal{G}_1 = \left( \int_{-1}^1 \varphi_i \varphi_j \, dx \right)_{i,j=1}^3 = \frac{1}{15} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},
\]

\[
\mathcal{G}_2 = \left( \int_{-1}^1 \psi_i \varphi_j \, dx \right)_{i,j=1}^3 = \frac{1}{54} \begin{bmatrix} 18 - \sqrt{3} & 36 - 16\sqrt{3} & -\sqrt{3} \\ 2\sqrt{3} & 32\sqrt{3} & 2\sqrt{3} \\ -\sqrt{3} & 36 - 16\sqrt{3} & 18 - \sqrt{3} \end{bmatrix}.
\]

It is easy to verify that \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \otimes \mathcal{G}_2^T \otimes \mathcal{G}_2^T \) are symmetric and positive definite. Thus \( \| \hat{u}_h \|_{0, Q}^2 \) and \( \int_Q \hat{u}_h \hat{u}_h \hat{I}^* h \hat{u}_h \hat{x} \) are equivalent. Applying (4.5) and (4.6) and summing the result over \( \Omega_h \) yield estimate (4.3). Estimate (4.4) was proved in Yang (2006).

We shall often use the Bramble–Hilbert lemma (Brenner & Scott, 2008) as recapped below.

**Lemma 4.2 (Bramble–Hilbert lemma)** Let \( L(u) \) be a continuous linear functional on \( W^{m,p}(\Omega) \) and \( W^{m,p}(\Omega)' \) be its dual norm. If \( L(u) \) vanishes for all \( u \in P^{m-1} \), then there exists a constant \( C = C(\Omega) \) such that

\[
|L(u)| \leq C|L|_{W^{m,p}(\Omega)} |u|_{W^{m,p}(\Omega)}. \tag{4.7}
\]

### 4.2 Error estimates for the elliptic problem

The bilinear form \( a_h(u, I_h^* v_h) \) can be rewritten as

\[
a_h(u, I_h^* v_h) = \sum_{Q \in \Omega_h} a_{Q,h}(u, I_h^* v_h) = \sum_{Q \in \Omega_h} \left( - \sum_{z \in Z \cap Q} v_h(z) \int_{\partial V \cap Q} \nabla u \cdot n \, ds \right) \\
= \sum_{Q \in \Omega_h} \left( \sum_{z_1, z_2 \in Z \cap Q} (v_h(z_1) - v_h(z_2)) \int_{\partial V_{z_1} \cap \partial V_{z_2}} \nabla u \cdot n \, ds \right), \tag{4.8}
\]

where \( z_1, z_2 \) are chosen in \( Q \) with no repetition. Applying the affine transformation \( F_Q \), we have

\[
\int_{\partial V_{z_1} \cap \partial V_{z_2}} \nabla u \cdot n \, ds = - \int_{\partial V_{z_1} \cap \partial V_{z_2}} (\hat{u}_1 b_{11} + \hat{u}_2 b_{12}) \, d\hat{y} + \int_{\partial V_{z_1} \cap \partial V_{z_2}} (\hat{u}_3 b_{21} + \hat{u}_4 b_{22}) \, d\hat{x}, \tag{4.9}
\]
where
\[ b_{11} = J_{F_0}^{-1} \begin{pmatrix} \frac{\partial x}{\partial y} \end{pmatrix}^2 + \begin{pmatrix} \frac{\partial y}{\partial y} \end{pmatrix}^2, \]
\[ b_{12} = b_{21} = -J_{F_0}^{-1} \begin{pmatrix} \frac{\partial x}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial y}{\partial x} \frac{\partial y}{\partial y} \end{pmatrix}, \]
\[ b_{22} = J_{F_0}^{-1} \begin{pmatrix} \frac{\partial x}{\partial x} \end{pmatrix}^2 + \begin{pmatrix} \frac{\partial y}{\partial y} \end{pmatrix}^2. \]

Differentiating the identity \( J_{F_0} J_{F_0}^{-1} = 1 \) and applying math induction, we have \( D^a J_{F_0}^{-1} = O(h_Q^{-2+|a|}). \) Then the estimate (Zlámal, 1978)
\[ |D^a b_{ij}| \lesssim h_Q^{|a|}, \quad \alpha \geq 0, \quad i, j = 1, 2 \tag{4.10} \]
follows from the Leibniz rule.

Next we prove a superapproximation result for \( a_h(u - I_h u, I_h^* v_h). \)

**Lemma 4.3** Assume that \( \Omega_h \) satisfies Mesh Assumptions A, B and C. If \( u \in H^4(\Omega) \) and \( v_h \in S_h, \) then
\[ |a_h(u - I_h u, I_h^* v_h)| \lesssim h^3 \| u \|_4 \| v_h \|_1. \tag{4.11} \]

**Proof.** Let \( w = u - I_h u. \) Using (4.8) and (4.9) gives
\[ a_h(w, I_h^* v_h) = \sum_{Q \in \Omega_h} a_{Q,h}(w, I_h^* v_h) \]
with
\[ a_{Q,h}(w, I_h^* v_h) = - \sum_{\hat{z}_1, \hat{z}_2 \in Z_h \cap \hat{Q}} (\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)) \int_{\phi_{V_1} \cap \phi_{V_2}} (\hat{w}_h b_{11} + \hat{w}_h b_{12}) \, d\hat{y} \]
\[ + \sum_{\hat{z}_1, \hat{z}_2 \in Z_h \cap \hat{Q}} (\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)) \int_{\phi_{V_1} \cap \phi_{V_2}} (\hat{w}_h b_{21} + \hat{w}_h b_{22}) \, d\hat{x}. \tag{4.12} \]

We only need to estimate the first term which includes the line integral along the \( \hat{y} \) direction. The integral along the \( \hat{x} \) direction can be estimated similarly.

Note that \( \hat{x} = \pm \sqrt{3}/3 \) in the first integral. By estimate (2.3) and the definition of the interpolation \( I_h, \) we have \( \hat{w}_h = 0 \) for \( \hat{u} = \hat{x}^j \hat{y}^i, 0 \leq i + j \leq 3. \) Hence, the Bramble–Hilbert lemma and (4.10) yield
\[ - \sum_{\hat{z}_1, \hat{z}_2 \in Z_h \cap \hat{Q}} (\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)) \int_{\phi_{V_1} \cap \phi_{V_2}} \hat{w}_h b_{11} \, d\hat{y} \lesssim |\hat{u}|_{4, \hat{Q}} \sum_{\hat{z}_1, \hat{z}_2 \in Z_h \cap \hat{Q}} |\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)| \]
\[ \lesssim \sum_{\hat{z}_1, \hat{z}_2 \in Z_h \cap \hat{Q}} h_Q^3 \| u \|_{4, \hat{Q}} |v_h(z_1) - v_h(z_2)| \lesssim h^3 \| u \|_{4, \hat{Q}} \| v_h \|_{1, \hat{h}, \hat{Q}}. \tag{4.13} \]
The estimation for

\[-\sum_{\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}_n \cap \tilde{Q}} (\hat{v}_h(\tilde{z}_1) - \hat{v}_h(\tilde{z}_2)) \int_{a_{V_{i_1}} \cap a_{V_{i_2}}} \hat{w}_y b_{12} \, d\tilde{y}\]

is technically involved. A starting point is to use (2.3) to get a superconvergence. But note that \(\hat{y} \in [-1, 1]\) and \(\hat{w}_y\) vanishes only for \(\hat{u} = \hat{x}^i \hat{y}^j, 0 \leq i, j \leq 2\). So, we shall construct a quadrature formula, in which \(\hat{y}\) is fixed at \(\pm \sqrt{3}/3\), to approximate the integral. Then applying Mesh Assumption C, we find that the lower order parts of the sum of the truncation terms on all elements can be cancelled on the element edges.

For each element \(Q\), let \(b_{12}\) be the average value of \(b_{12}\). By (4.10), we have \(b_{12} = \bar{b}_{12} + \mathcal{O}(h_Q)\). Then

\[-\sum_{\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}_n \cap \tilde{Q}} (\hat{v}_h(\tilde{z}_1) - \hat{v}_h(\tilde{z}_2)) \int_{a_{V_{i_1}} \cap a_{V_{i_2}}} \hat{w}_y b_{12} \, d\tilde{y}\]

\[\leq -\bar{b}_{12} \sum_{\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}_n \cap \tilde{Q}} (\hat{v}_h(\tilde{z}_1) - \hat{v}_h(\tilde{z}_2)) \int_{a_{V_{i_1}} \cap a_{V_{i_2}}} \hat{w}_y \, d\tilde{y} + C h^3 \|u\|_3, Q |v_h|_{1, h, Q}. \tag{4.14}\]

We introduce a quadrature formula to approximate \(\int_{a_{V_{i_1}} \cap a_{V_{i_2}}} \hat{w}_y \, d\tilde{y}\). For any function \(f(\hat{y})\),

\[\int_{-\sqrt{3}/3}^{\sqrt{3}/3} f \, d\hat{y} \approx \int_{-\sqrt{3}/3}^{\sqrt{3}/3} f \left( \frac{-\sqrt{3}}{3} \right) \, d\hat{y}, \quad \int_{\sqrt{3}/3}^{1} f \, d\hat{y} \approx \int_{\sqrt{3}/3}^{1} f \left( \frac{\sqrt{3}}{3} \right) \, d\hat{y}\]

and

\[\int_{-\sqrt{3}/3}^{\sqrt{3}/3} f \, d\hat{y} \approx \frac{1}{2} \int_{-\sqrt{3}/3}^{\sqrt{3}/3} \left( f \left( \frac{-\sqrt{3}}{3} \right) + f \left( \frac{\sqrt{3}}{3} \right) \right) \, d\hat{y}.\]

Denoting by \(\int_{a_{V_{i_1}} \cap a_{V_{i_2}}} \hat{w}_y \, d\tilde{y}\) the approximation of \(\int_{a_{V_{i_1}} \cap a_{V_{i_2}}} \hat{w}_y \, d\tilde{y}\). Noting that \(\hat{y} = \pm \sqrt{3}/3\) in \(\hat{w}_y\), we have

\[-\bar{b}_{12} \sum_{\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}_n \cap \tilde{Q}} (\hat{v}_h(\tilde{z}_1) - \hat{v}_h(\tilde{z}_2)) \int_{a_{V_{i_1}} \cap a_{V_{i_2}}} \hat{w}_y \, d\tilde{y} \leq -\bar{b}_{12} R_1 + C h^3 \|u\|_4, Q |v_h|_{1, h, Q}. \tag{4.15}\]

where

\[R_1 = \sum_{\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}_n \cap \tilde{Q}} (\hat{v}_h(\tilde{z}_1) - \hat{v}_h(\tilde{z}_2)) \int_{a_{V_{i_1}} \cap a_{V_{i_2}}} (\hat{w}_y^{\text{app}} - \hat{w}_y) \, d\tilde{y}.\]

To estimate term \(R_1\), we construct an auxiliary functional

\[F_1 = \frac{\sqrt{3}}{18} \left\{ \int_{-1}^{1} \frac{\partial^2 \hat{w}(1, \hat{y})}{\partial \hat{y}^2} \frac{\partial \hat{v}_h(1, \hat{y})}{\partial \hat{y}} \, d\hat{y} - \int_{-1}^{1} \frac{\partial^2 \hat{w}(-1, \hat{y})}{\partial \hat{y}^2} \frac{\partial \hat{v}_h(-1, \hat{y})}{\partial \hat{y}} \, d\hat{y} \right\}.\]

Set \(L(\hat{u}) = R_1 + F_1\). If \(\hat{u} = \hat{x}^i \hat{y}^j, 0 \leq i, j \leq 2\), then \(\hat{w} = 0\). If \(\hat{u} = \hat{x}^3\), then \(\hat{w}_y = 0\). If \(\hat{u} = \hat{y}^3\), a direct calculation produces

\[R_1 \big|_{\hat{u}=\hat{x}^3} = -F_1 \big|_{\hat{u}=\hat{x}^3} = -\frac{4\sqrt{3}}{9} \frac{\partial^3 \hat{v}_h}{\partial x \partial y^2} (0, 0).\]
Thus $L(\hat{u})$ vanishes for $\hat{u} = \hat{\chi}'\hat{\psi}'$, $0 \leq i + j \leq 3$. Note that $L(\hat{u})$ is a linear functional of $\hat{u}$. By the Bramble–Hilbert lemma and the norm equivalence in the finite-dimensional spaces, we have

$$L(\hat{u}) \lesssim \|\hat{u}\|_{4,\Omega}(|\hat{\nu}_h(\hat{z}_1)| + |\hat{\nu}_{\hat{\eta}}(\hat{z}_2)| + \|\hat{\nu}_h\|_{H^1(\partial\hat{\Omega})})$$

$$\lesssim \|\hat{u}\|_{4,\Omega}(|\hat{\nu}_h|_{1,h,\hat{\Omega}} + |\hat{\nu}_{\hat{\eta}}|_{1,\hat{\Omega}}) \lesssim h^3\|u\|_{4,\Omega}|v_h|_{1,\hat{\Omega}}. \quad (4.16)$$

Since $R_1 = L(\hat{u}) - F_1$, combining (4.14) to (4.16) together yields

$$- \sum_{\hat{z}_1,\hat{z}_2 \in \hat{\Omega} \cap Q} (\hat{\nu}_h(\hat{z}_1) - \hat{\nu}_h(\hat{z}_2)) \int_{\partial\hat{\Omega} \cap \partial\Omega} \hat{w}_h \hat{b}_{12} d\hat{y} \lesssim h^3\|u\|_{4,\Omega}|v_h|_{1,\hat{\Omega}} - \hat{b}_{12} F_1. \quad (4.17)$$

The estimation for the second integral in (4.12) is similar. Hence,

$$a_{Q,h}(w, I^*_h v_h) \lesssim h^3\|u\|_{4,\Omega}(|v_h|_{1,h,\hat{\Omega}} + |v_h|_{1,\hat{\Omega}}) - \hat{b}_{12} F_1 - \hat{b}_{21} F_2,$$

where $F_2$ is a functional similar to $F_1$, with $\hat{y}$ being replaced by $\hat{x}$. Therefore,

$$|a_h(w, I^*_h v_h)| \lesssim h^3\|u\|_{4,\Omega}(|v_h|_{1,h} + |v_h|_{1}) + \sum_{Q \in \hat{\Omega}_h} \hat{b}_{12} F_1 + \sum_{Q \in \Omega_h} \hat{b}_{21} F_2. \quad (4.18)$$

The sum $\sum_{Q \in \hat{\Omega}_h} \hat{b}_{ij} F_i(\hat{u})$ involves integrals on the edges of $Q$. On $\partial Q \cap \partial \Omega$, the corresponding integrals vanish because $v_h|_{\partial \Omega} = 0$. On an interior edge $\partial Q \cap \partial Q'$, the integrals from $Q$ and $Q'$ are taken for the same integrand but in opposite directions. Thus we can regroup the sum by edges to obtain the following terms on the interior edges

$$\frac{\sqrt{3}}{18}(\hat{b}_{12} - \hat{b}_{12}') \int_{-1}^{1} \frac{\partial^2\hat{\nu}_h(\hat{\gamma}, \hat{\psi})}{\partial \hat{\gamma}^2} \frac{\partial \hat{\nu}_h(\hat{\gamma}, \hat{\psi})}{\partial \hat{\psi}} d\hat{\gamma}, \quad \frac{\sqrt{3}}{18}(\hat{b}_{21} - \hat{b}_{21}') \int_{-1}^{1} \frac{\partial^2\hat{\nu}_h(\hat{\gamma}, \hat{\psi})}{\partial \hat{\gamma}^2} \frac{\partial \hat{\nu}_h(\hat{\gamma}, \hat{\psi})}{\partial \hat{\psi}} d\hat{\gamma}. \quad (4.17)$$

Under our mesh assumptions, $|\hat{b}_{ij} - \hat{b}_{ij}'|$ is $O(h)$ (see, e.g., Zlámal, 1978). Therefore, it follows from the Bramble–Hilbert lemma that

$$\left| \sum_{Q \in \hat{\Omega}_h} \hat{b}_{12} F_1(\hat{u}) \right| + \left| \sum_{Q \in \Omega_h} \hat{b}_{21} F_2(\hat{u}) \right|$$

$$\lesssim h \sum_{Q \in \hat{\Omega}_h} \left( \int_{-1}^{1} \frac{\partial^2\hat{\nu}_h(\hat{\gamma}, \hat{\psi})}{\partial \hat{\gamma}^2} \frac{\partial \hat{\nu}_h(\hat{\gamma}, \hat{\psi})}{\partial \hat{\psi}} d\hat{\gamma} \right)$$

$$\lesssim h \sum_{Q \in \hat{\Omega}_h} \|\hat{u}\|_{3,\hat{\Omega}}|\hat{\nu}_h|_{1,\hat{\Omega}} \lesssim h^3\|u\|_{3,\hat{\Omega}}|v_h|_{1}. \quad (4.19)$$

Combining (4.18) and (4.19) together and using the norm equivalence give the desired result.

**Lemma 4.4** Assume that $\Omega_h$ satisfies Mesh Assumptions A and B. Then

$$|a_h(u_h, I^*_h v_h)| \lesssim |u_h| |v_h|, \quad \forall u_h, v_h \in S_h, \quad (4.20)$$

$$a_h(u_h, I^*_h u_h) \geq |u_h|^2, \quad \forall u_h \in S_h. \quad (4.21)$$
Proof. The continuity (4.20) has been proved in Yang & Liu (2011). The coercivity (4.21) holds when \(|\cos \theta_Q| < 0.99\). The proof is to decompose the distortion of affine mappings into contraction and rotation and to employ partitioned matrices for size reduction, which is the same as that for Lemma 3.8 in Yang & Liu (2011) and is also similar to that for Lemma 4.5 in this paper. □

Theorem 4.1 Let \( u \) be the solution of (3.1) and \( u_h \) the finite-volume solution of (3.7). Assume that \( u \in H^4(\Omega) \cap H^1_0(\Omega) \). If Mesh Assumptions A, B and C are satisfied, then
\[
\|u - u_h\|_0 + h|u - u_h|_1 + |I_h u - u_h|_1 \lesssim h^3\|u\|_4.
\] (4.22)

Proof. We decompose the error as \( u_h - u = \xi - \eta \), where \( \xi = u_h - I_h u \) and \( \eta = u - I_h u \). By (3.5), (3.7), and Lemma 4.4, we have
\[
|\xi|_1^2 \lesssim a_h(\xi, I_h^* \xi) = a_h(\eta, I_h^* \xi).
\]
It follows from Lemma 4.3 that
\[
a_h(\eta, I_h^* \xi) \lesssim h^3\|u\|_4|\xi|_1.
\]
Therefore,
\[
|\xi|_1 \lesssim h^3\|u\|_4.
\]
The approximation property (2.2) and a triangle inequality lead to
\[
|u - u_h|_1 \leq |\xi|_1 + |\eta|_1 \lesssim h^2\|u\|_4
\]
and
\[
\|u - u_h\|_0 \leq |\xi|_0 + |\eta|_0 \lesssim |\xi|_1 + |\eta|_0 \lesssim h^3\|u\|_4,
\]
which complete the proof. □

Remark 4.1 We have obtained a superapproximation result for \( |I_h u - u_h|_1 \) in Theorem 4.1. This superconvergence not only helps us to derive optimal-order error estimates in the \( L^2 \)-norm, but can also be used for developing a posteriori error estimators or postprocessing algorithms, which will be pursued in our future work. The superconvergence is based on the Barlow points and Mesh Assumptions A, B and C. If only Mesh Assumptions A and B are considered, then the optimal order \( L^2 \) error estimate can also be obtained by using a duality argument and comparison with the finite-element bilinear forms.

4.3 Error estimates for the parabolic problem

According to Lemma 4.1, utilizing the Barlow points results in the term
\[
\int_{\hat{Q}} \hat{u}_h I_h^* v_h \hat{d}x = v_{\hat{Q}}^T \mathcal{F}_2 \otimes \mathcal{F}_2 u_Q
\] (4.23)
being nonsymmetric, since \( \mathcal{F}_2 \) is nonsymmetric. We intend to ‘symmetrize’ the above bilinear form on the reference element.
Lemma 4.5. For any $u_h, v_h \in S_h$, there exist $\tilde{u}_h, \tilde{v}_h \in S_h$ such that

$$\int_{\tilde{Q}} \tilde{u}_h^T \tilde{v}_h \, d\tilde{x} = \int_{\tilde{Q}} \tilde{v}_h^T \tilde{u}_h \, d\tilde{x},$$  \hspace{1cm} (4.24)

$$(u_h, I_h^e u_h)^{1/2} \sim \|u_h\|_0,$$  \hspace{1cm} (4.25)

$$|\tilde{u}_h|_{1,h} \sim |u_h|_{1,h},$$  \hspace{1cm} (4.26)

$$a_h(u_h, I_h^e u_h) \geq |u_h|^2.$$  \hspace{1cm} (4.27)

Proof. The proof consists of three major steps.

The first step is to construct $\tilde{u}_h$ for any $u_h \in S_h$. Assume that $\tilde{u}_h$ has the form $\tilde{u}_h^T = u_h^T \mathcal{Q} \otimes \mathcal{P}$. Utilizing a computer algebra system, e.g., Matlab, we obtain (generally not unique though)

$$\mathcal{P} = \frac{1}{6} \begin{bmatrix} 6 & 3 - 2\sqrt{3} & 0 \\ 0 & 4\sqrt{3} & 0 \\ 0 & 3 - 2\sqrt{3} & 6 \end{bmatrix}. $$

The second step is to verify (4.24–4.26). By the proof of Lemma 3.1, we have

$$\int_{\tilde{Q}} \tilde{u}_h^T \tilde{v}_h \, d\tilde{x} = \sum_{Q \in \mathcal{O}_h} \tilde{v}_h^T \mathcal{Q} \mathcal{G}_2 \otimes \mathcal{P} u_Q = \sum_{Q \in \mathcal{O}_h} (\mathcal{G}_2 \mathcal{Q} \mathcal{P}) u_Q = \sum_{Q \in \mathcal{O}_h} \mathcal{G}_2 \mathcal{Q} (\mathcal{P} \mathcal{G}_2) u_Q. \hspace{1cm} (4.28)$$

Let

$$\mathcal{G}_2 = \mathcal{P} \mathcal{G}_2 = \frac{1}{27} \begin{bmatrix} 8 & 2 & -1 \\ 2 & 32 & 2 \\ -1 & 2 & 8 \end{bmatrix}. $$

Then (4.24) follows by the symmetry of $\mathcal{G}_2$. On the other hand, according to Lemma 4.1, in order to prove (4.25), we only need to prove that $\int_{\tilde{Q}} \tilde{u}_h^T \tilde{v}_h \, d\tilde{x}$ is a quadratic form on $u_Q$, which is straightforward by the positive definiteness of matrix $\mathcal{G}_2$.

For any $u_h \in S_h$, let $\alpha = (u_{x,Q}, u_{y,Q})$ be a vector with $u_{x,Q}, u_{y,Q} \in \mathbb{R}^6$ defined as

$$u_{x,Q} = (\delta_x u_h(x_{i+1/2,j}), \delta_x u_h(x_{i+1/2,j+1/2}), \delta_x u_h(x_{i+1,j+1}), ... , \delta_x u_h(x_{i+1,j+1})), $$

$$u_{y,Q} = (\delta_y u_h(x_{i,j+1/2}), \delta_y u_h(x_{i+1/2,j+1/2}), \delta_y u_h(x_{i+1,j+1/2}), ... , \delta_y u_h(x_{i+1,j+1})). $$

Since $\tilde{u}_Q^T = u_Q^T \mathcal{Q} \otimes \mathcal{P}$, a direct calculation yields

$$\tilde{u}_{x,Q}^T = u_{x,Q}^T \mathcal{P} \otimes \mathcal{Q}, \hspace{1cm} \tilde{u}_{y,Q}^T = u_{y,Q}^T \mathcal{Q} \otimes \mathcal{P}, \hspace{1cm} (4.29)$$

where

$$\mathcal{Q} = \frac{1}{6} \begin{bmatrix} 3 + 2\sqrt{3} & 3 - 2\sqrt{3} \\ 3 - 2\sqrt{3} & 3 + 2\sqrt{3} \end{bmatrix}. $$
Noting that $\mathcal{P}$ and $\mathcal{D}$ are nonsingular, we have
\[
|\tilde{u}_h|_{1,h,Q}^2 = (\tilde{u}_{x,Q}^T \tilde{u}_{x,Q} + \tilde{u}_{y,Q}^T \tilde{u}_{y,Q}) \sim (u_{x,Q}^T u_{x,Q} + u_{y,Q}^T u_{y,Q}) = |u_h|_{1,h,Q}^2.
\]

The third step is to prove (4.27), which uses Lemma 4.6 about partitioned matrices that was proved in Yang (2006).

Let $P_1$, $P_2$ be two points. We use $|P_1P_2|$ to denote its length. Without loss of generality, we choose $\theta_Q \neq 0$ and suppose that $Q$ is a parallelogram with two edges $P_1P_2$ and $P_1P_4$ (see Fig. 3). According to Lemma 3.6 in Yang & Liu (2011), under Mesh Assumptions A and B, the difference between the bilinear form $a_h(u_h, I_h^* v_h)$ on a quadrilateral and the one on a parallelogram is $O(h)|u_h||v_h|$. In order to prove (4.27), we only need to prove that $a_{Q,h}(u_h, I_h^* \tilde{u}_h) \geq |u_h|_{1,Q}^2$ holds on the parallelogram.

Let $\kappa = |P_1P_4|/|P_1P_2|$. By (4.9), we can derive
\[
a_{Q,h}(u_h, I_h^* \tilde{u}_h) = \tilde{\mathcal{A}} \mathcal{A}^T = \mathcal{J} \mathcal{A} \mathcal{A}^T, \quad (4.30)
\]
where
\[
\mathcal{J} = \begin{bmatrix} \mathcal{P} \otimes \mathcal{D} & \mathcal{P} \otimes \mathcal{D} \end{bmatrix}, \quad \mathcal{A} = \frac{|P_1P_2|^2}{m(Q)} \begin{bmatrix} \mathcal{A}_1 & \kappa \mathcal{A}_2 \\ \kappa \mathcal{A}_2 & \kappa^2 \mathcal{A}_1 \end{bmatrix}
\]
with
\[
\mathcal{A}_1 = \frac{1}{6} \begin{bmatrix} 3 + 2\sqrt{3} & 3 - 2\sqrt{3} \\ 3 - 2\sqrt{3} & 3 + 2\sqrt{3} \end{bmatrix} \otimes \frac{1}{54} \begin{bmatrix} 18 - \sqrt{3} & 36 - 16\sqrt{3} & -\sqrt{3} \\ 2\sqrt{3} & 32\sqrt{3} & 2\sqrt{3} \\ -\sqrt{3} & 36 - 16\sqrt{3} & 18 - \sqrt{3} \end{bmatrix},
\]
\[
\mathcal{A}_2 = \frac{\cos \theta_Q}{18} \begin{bmatrix} \mathcal{A}_{21} & \mathcal{A}_{22} \\ \mathcal{A}_{23} & \mathcal{A}_{24} \end{bmatrix},
\]
\[
\mathcal{A}_{21} = \begin{bmatrix} -2\sqrt{3} - 1 & 2\sqrt{3} - 10 & 3\sqrt{3} - 4 \\ -\sqrt{3} - 3 & -4\sqrt{3} & -\sqrt{3} + 3 \\ 1 & 2\sqrt{3} - 2 & \sqrt{3} - 2 \end{bmatrix}, \quad \mathcal{R} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]
\[
\mathcal{A}_{22} = \mathcal{R} \mathcal{A}_{21}, \quad \mathcal{A}_{23} = \mathcal{A}_{21} \mathcal{R}, \quad \mathcal{A}_{24} = \mathcal{R} \mathcal{A}_{21} \mathcal{R}.
\]
Let
\[ \mathcal{A} = \mathcal{I} \mathcal{A} = \frac{|P_1 P_2|^2}{m(Q)} \begin{bmatrix} \mathcal{P} \otimes \mathcal{A}_1 \kappa \mathcal{P} \otimes \mathcal{A}_2 \kappa^2 \mathcal{P} \otimes \mathcal{A}_1 \end{bmatrix} = \frac{|P_1 P_2|^2}{m(Q)} \begin{bmatrix} \mathcal{A}_1 \kappa \mathcal{A}_2 \kappa \mathcal{A}_2 \kappa^2 \mathcal{A}_1 \end{bmatrix}, \]
where \( m(Q) \) is the measure of \( Q \). We calculate all the sequential principal minors of the matrices
\[ \left( \frac{\mathcal{A}_1 + \mathcal{A}_1^T}{2} \pm \frac{\mathcal{A}_2 + \mathcal{A}_2^T}{2} \right) \]
and find that the principal minors of the above matrices are positive when \(|\cos(\theta_Q)| < 0.97\). Hence the matrices \((\mathcal{A}_1 + \mathcal{A}_1^T)/2 \pm (\mathcal{A}_2 + \mathcal{A}_2^T)/2\) are positive definite. By Lemma 4.6, we know that the matrix \((\mathcal{A}_1 + \mathcal{A}_1^T)/2\) is also positive definite. Its smallest eigenvalue \( \lambda_\gamma \) depends only on the constant \( \gamma \). From Mesh Assumption A, we have
\[ \lambda_{\min} \left( \frac{\mathcal{A}_1 + \mathcal{A}_1^T}{2} \pm \frac{\mathcal{A}_2 + \mathcal{A}_2^T}{2} \right) \geq \lambda_\gamma \frac{|P_1 P_2|^2}{m(Q)} \geq \frac{2 \lambda_\gamma}{\sigma}. \] (4.31)
Accordingly the following holds for a parallelogram
\[ a_{Q,h}(u_h, I_h^* \tilde{u}_h) = a \left( \mathcal{I} \mathcal{A} + (\mathcal{I} \mathcal{A})^T \right) \frac{\alpha^T}{2} a \geq a \left( \frac{\mathcal{A}_1 + \mathcal{A}_1^T}{2} \right) \frac{\alpha^T}{2} a \geq \frac{2 \lambda_\gamma}{\sigma} |u_h|^2_{1,h,Q}. \]
Therefore, \( a_h(u_h, I_h^* \tilde{u}_h) \gtrsim |u_h|^2_{1} \) holds for quadrilateral meshes that satisfy Mesh Assumptions A and B.

**Lemma 4.6** The matrix \( \begin{bmatrix} \mathcal{A} & \kappa \mathcal{A} \kappa \mathcal{A} \kappa^2 \mathcal{A} \end{bmatrix} \) is positive definite if and only if matrices \( \mathcal{A} \pm \frac{1}{2}(\mathcal{A} + \mathcal{A}^T) \) are positive definite.

**Lemma 4.7** Assume Mesh Assumptions A and B are satisfied. For any \( u_h, v_h \in S_h \), we have
\[ |(u_h, I_h^* \tilde{v}_h) - (v_h, I_h^* \tilde{u}_h)| \lesssim h \|u_h\|_0 \|v_h\|_0. \] (4.32)

**Proof.** For any \( Q \in \Omega_h \), a change of variables in multiple integrals gives us
\[ \int_Q u_h I_h^* \tilde{v}_h \, d\mathbf{x} = \int_{\tilde{Q}} J_{F_Q} \hat{u}_h I_{\tilde{Q}}^* \tilde{v}_h \, d\tilde{\mathbf{x}}. \]
By (4.24), we have
\[ J_{F_Q}(0, 0) \int_{\tilde{Q}} \hat{u}_h I_{\tilde{Q}}^* \tilde{v}_h \, d\tilde{\mathbf{x}} - J_{F_Q}(0, 0) \int_{\tilde{Q}} \hat{v}_h I_{\tilde{Q}}^* \tilde{u}_h \, d\tilde{\mathbf{x}} = 0. \] (4.33)

It was proved in Li & Li (1999) under Mesh Assumptions A and B that
\[ |J_{F_Q} - J_{F_Q}(0, 0)| \lesssim h_Q^3. \]
Combining the above estimates and using (4.28), we obtain
\[
\left| \int_Q u_h I_h^* \tilde{v}_h \, dx - \int_Q v_h I_h^* \tilde{u}_h \, dx \right| \leq \left| \int_Q u_h I_h^* \tilde{v}_h \, dx - J_{FQ}(0,0) \int_Q \tilde{u}_h I_h^* \tilde{v}_h \, dx \right| \\
+ \left| \int_Q v_h I_h^* \tilde{u}_h \, dx - J_{FQ}(0,0) \int_Q \tilde{v}_h I_h^* \tilde{u}_h \, dx \right| \\
\lesssim h^3 \| u_h \|_{0,Q} \| \tilde{u}_h \|_{0,Q} \lesssim h \| u_h \|_{0,Q} \| v_h \|_{0,Q}.
\]

Therefore, by the Cauchy–Schwarz inequality,
\[
| (u_h, I_h^* \tilde{v}_h) - (v_h, I_h^* \tilde{u}_h) | \leq \sum_{Q \in \Omega_h} \left| \int_Q u_h I_h^* \tilde{v}_h \, dx - \int_Q v_h I_h^* \tilde{u}_h \, dx \right| \\
\lesssim \sum_{Q \in \Omega_h} h_Q \| u_h \|_{0,Q} \| v_h \|_{0,Q} \lesssim h \| u_h \|_{0,Q} \| v_h \|_{0,Q}.
\]

\[\text{THEOREM 4.2} \quad \text{Let} \ u \text{ be the solution of (3.8) and } u_h \text{ the numerical solution of the semidiscrete finite-volume scheme (3.9). Assume that } u \in L^\infty(0,T;H^1) \cap H^1(0,T;H^1) \text{ and Mesh Assumptions A, B and C are satisfied. Then}
\]
\[
\| u(t) - u_h(t) \|_0 + \| u(t) - u_h(t) \|_1 + \left( \int_0^t \| I_h u - u_h \|_1^2 \, dt \right)^{1/2} \lesssim h^3, \quad 0 \leq t \leq T. \tag{4.34}
\]

\[\text{Proof.} \quad \text{We decompose the error as } u_h - u = \xi - \eta, \text{ where } \xi = u_h - R_h u \text{ and } \eta = u - R_h u. \text{ We have the following error equation:}
\]
\[
(\xi_t, I_h^* v_h) + a_h(\xi, I_h^* v_h) = (\eta_t, I_h^* v_h). \tag{4.35}
\]

Taking \( v_h = \xi_t \) in (4.35) leads to
\[
\| \xi_t \|_0^2 = (\eta_t, I_h^* \xi_t) - a_h(\xi, I_h^* \xi_t) \\
\lesssim \| \eta_t \|_0 \| I_h^* \xi_t \|_0 + C \| \xi \|_1 \| \xi_t \|_1 \\
\lesssim \| \eta_t \|_0 \| I_h^* \xi_t \|_0 + Ch^{-1} \| \xi \|_1 \| \xi_t \|_0,
\]
where an inverse estimate is used for the last inequality. Utilizing the norm equivalence, we have
\[
\| \xi_t \|_0 \lesssim \| \eta_t \|_0 + h^{-1} \| \xi \|_1. \tag{4.36}
\]

On the other hand, we take \( v_h = \tilde{\xi} \) in (4.35) to obtain
\[
\frac{1}{2} \frac{d}{dt} (\xi, I_h^* \tilde{\xi}) + a_h(\xi, I_h^* \tilde{\xi}) = \frac{1}{2} \left( (\xi_t, I_h^* \tilde{\xi}) - (\eta_t, I_h^* \tilde{\xi}) \right) + (\eta_t, I_h^* \tilde{\xi}).
\]
By Lemmas 4.5 and 4.7, we have
\[
\frac{1}{2} \frac{d}{dt} (\xi, I_h^* \tilde{\xi}) + |\xi|^2 \lesssim h \|\xi\|_0 \|\xi_t\|_0 + \|\eta_t\|_0 \|I_h^* \tilde{\xi}\|_0
\lesssim h \|\xi\|_0 \|\xi_t\|_0 + \|\eta_t\|_0 \|\xi\|_0.
\] (4.37)

Combining (4.36) and (4.37) together yields
\[
\frac{1}{2} \frac{d}{dt} (\xi, I_h^* \tilde{\xi}) + |\xi|^2 \lesssim h \|\xi\|_0 \|\xi_t\|_0 + \|\eta_t\|_0 \|\eta\|_0 \|\xi\|_0 + \|\eta_t\|_0 \|\xi\|_0.
\] (4.38)

Integrating (4.38) on \([0, t]\), noting that \(\xi(0) = 0\), and using Lemma 4.5, we have
\[
\|\xi(t)\|_0^2 + \int_0^t |\xi|^2 \, dt \lesssim \int_0^t ((h + 1) \|\xi\|_0 \|\eta_t\|_0 + \|\xi\|_0 |\xi|_1) \, dt
\lesssim C \int_0^t \|\xi\|_0^2 \, dt + C \int_0^t \|\eta_t\|_0^2 \, dt + \epsilon \int_0^t |\xi|^2 \, dt.
\] Then taking \(\epsilon < 1\) and using the Gronwall’s inequality yield
\[
\|\xi(t)\|_0^2 + \int_0^t |\xi|^2 \, dt \lesssim \int_0^t \|\eta_t\|_0^2 \, dt.
\] (4.39)

By the inverse estimate,
\[
h |\xi|_1 \lesssim \|\xi\|_0 \lesssim \left(\int_0^t \|\eta_t\|_0^2 \, dt\right)^{1/2}.
\] (4.40)

From (4.39), (4.40) and Theorem 4.1, we have
\[
\|u - u_h\|_0 + h |u - u_h|_1 + \left(\int_0^t |I_h u - u_h|_1^2 \, dt\right)^{1/2}
\lesssim \|\xi\|_0 + \|u - R_h u\|_0 + h (|\xi|_1 + |u - R_h u|_1) + \left(2 \int_0^t |\xi|_1^2 + |I_h u - R_h u|_1^2 \, dt\right)^{1/2}
\lesssim h^3 \left(\|u\|_{L^\infty(H^4)} + \left(\int_0^t \|u_t\|_4^2 \, dt\right)^{1/2}\right),
\]
which gives the desired result.

**Theorem 4.3** Let \(u\) be the solution of (3.8) and \(u^n_h\) the numerical solution of the fully discrete finite-volume scheme (3.12). Assume that \(u \in L^\infty(0, T; H^4) \cap H^1(0, T; H^4) \cap H^3(0, T; L^2)\) and Mesh Assumptions A, B and C are satisfied. Then we have the following error estimate:
\[
\|u^M - u^M_h\|_0 + h |u^M - u^M_h|_1 + \left(\Delta t \sum_{n=1}^M |I_h u^{n,1/2} - u^{n,1/2}_h|_1^2\right)^{1/2}
\lesssim h^3 + \Delta t^2, \quad 0 \leq M \leq N.
\] (4.41)
Proof. Let \( \xi = u_h - R_h u \) and \( \eta = u - R_h u \). It is obvious that \( \xi^0 = 0 \). We have the following error equation:

\[
(\partial \xi^n, I_h^s v_h) + a_h(\xi^{n,1/2}, I_h^s v_h) = (\alpha^n, I_h^s v_h), \quad n \geq 1,
\]

where

\[
\alpha^n = \partial \eta^n + u_t^{n,1/2} - \partial u^n.
\]

We choose \( v_h = \tilde{\xi}^n \) in (4.42) and take a similar argument as that in Theorem 4.2 to obtain

\[
\|\tilde{\xi}^n\|_0 \lesssim \|\alpha^n\|_0 + h^{-1}|\xi^{n,1/2}|_1.
\]

On the other hand, we choose \( v_h = 2\Delta t\tilde{\xi}^{n,1/2} \) in (4.42) to obtain

\[
(\xi^n, I_h^s \tilde{\xi}^n) - (\xi^{n-1}, I_h^s \tilde{\xi}^{n-1}) + 2\Delta t a_h(\xi^{n,1/2}, I_h^s \tilde{\xi}^{n,1/2})
= (\xi^{n-1}, I_h^s \tilde{\xi}^{n-1}) - (\xi^n, I_h^s \tilde{\xi}^{n-1}) + 2\Delta t(\alpha^n, I_h^s \tilde{\xi}^{n,1/2}),
\]

where we use Lemma 4.7 and (4.43) to find

\[
(\xi^{n-1}, I_h^s \tilde{\xi}^{n-1}) - (\xi^n, I_h^s \tilde{\xi}^{n-1}) = (\xi^{n-1}, I_h^s (\tilde{\xi}^n - \tilde{\xi}^{n-1}) - (\xi^n - \xi^{n-1}, I_h^s \tilde{\xi}^{n-1})
\leq C\Delta t\|\tilde{\xi}^n\|_0\|\xi^{n-1}\|_0 \leq C\Delta t(\|\alpha^n\|_0 + |\xi^{n,1/2}|_1)\|\xi^{n-1}\|_0.
\]

Therefore, summing from \( n = 1 \) to \( M \) yields

\[
(\xi^M, I_h^s \tilde{\xi}^M) + 2\Delta t \sum_{n=1}^M a_h(\xi^{n,1/2}, I_h^s \tilde{\xi}^{n,1/2})
\leq C\Delta t\sum_{n=1}^M (h\|\alpha^n\|_0 + |\xi^{n,1/2}|_1)\|\xi^{n-1}\|_0 + 2\Delta t \sum_{n=1}^M (\alpha^n, I_h^s \tilde{\xi}^{n,1/2})
\leq C\Delta t\sum_{n=1}^M \|\alpha^n\|_0^2 + C\Delta t\sum_{n=1}^M \|\xi^{n-1}\|_0^2 + \epsilon\Delta t\sum_{n=1}^M |\xi^{n,1/2}|_1^2.
\]

Using Lemma 4.5, taking \( \epsilon \) small enough and applying the discrete Gronwall’s inequality, we have

\[
\|\xi^M\|_0^2 + \Delta t\sum_{n=1}^M |\xi^{n,1/2}|_1^2 \lesssim \Delta t\sum_{n=1}^M \|\alpha^n\|_0^2.
\]

The Taylor’s expansion with the remainder term in the integral form gives

\[
\Delta t \sum_{n=1}^M \|\alpha^n\|_0^2 \leq 2\Delta t \sum_{n=1}^M (\|\tilde{\partial} \eta^n\|_0^2 + u_t^{n,1/2} - \partial u^n \|_0^2)
\leq C \left( h^6 \int_0^{t_M} \|u_t\|_4^2 \, dt + \Delta t^4 \int_0^{t_M} \|u_{ttt}\|_0^2 \, dt \right).
\]
Hence,
\[ \| \xi^M \|_0^2 + \Delta t \sum_{n=1}^{M} | \xi^{n,1/2} |_1^2 \lesssim h^6 \int_0^{tM} \| u_t \|_4^2 \, dt + \Delta t^4 \int_0^{tM} \| u_{tt} \|_0^2 \, dt. \]

Finally, the desired result follows by an inverse estimate and a triangle inequality. \( \square \)

**Remark 4.2** Note from Theorems 4.1–4.3 that our error estimates are optimal with respect to the degree of approximating polynomials used, but suboptimal with respect to the solution regularity. There are two main reasons. First, to obtain optimal order errors, finite-volume element methods almost need higher regularity assumptions on the exact solution than the finite element methods (even for linear cases), see, e.g., Chou & Ye (2007), Xu & Zou (2009) and Yang (2006). Secondly, the analysis is based on the superapproximation result for \( a_h(u - I_h u, l_h^\ast v_h) \), which convergences one order higher than the optimal rate. The errors in the \( H^1 \)- and \( L^2 \)-norms can be regarded as a by-product of this superconvergence result. This technique successfully avoids the difficulty of the duality argument for higher order finite-volume methods but needs higher regularity on the exact solution.

5. Numerical experiments

In this section, we present numerical results to illustrate the theoretical findings in this paper.

We test the finite-volume methods for elliptic and parabolic problems on a family of quadrilateral meshes that are obtained by perturbing a set of rectangular meshes randomly. The rectangular meshes have \( M = 8, 16, 32, 64, 128 \) partitions, respectively, in both \( x, y \)-directions. The quadrilateral meshes are then generated by

\[ x_{ij} = \frac{\pi}{M} i + \frac{\pi}{4M} \sin(j) \text{rand}(), \quad y_{ij} = \frac{\pi}{M} j + \frac{\pi}{4M} \sin(i) \text{rand}(), \quad 1 \leq i, j \leq M - 1, \]

where \( \sin(i), \sin(j) \) use the radian unit for angles and \text{rand}() \) generates a random number in \((0, 1)\).

The preconditioned bi-conjugate gradient stabilized method (BiCGStab) is employed to solve the nonsymmetric discrete linear systems. The tolerance for residuals is set as \( 10^{-12} \) and the simple diagonal preconditioning is used. The numerical order of convergence is then measured by comparing the computed errors on two successive mesh levels.

**Example 1** (An Elliptic Problem) The domain is \( \Omega = [0, \pi]^2 \), the permeability coefficient \( a(x, y) = (x + 1)^2 + y^2 \), the exact solution \( u(x, y) = \sin(x) \sin(y) \). This problem was tested in Yang (2006).

Table 1 shows a third-order convergence in \( L^2 \)-norm of the error \( u - u_h \) and also a third-order convergence in the \( H^1 \)-seminorm of \( h(u - u_h) \). One can also observe a third-order superapproximation in \( |I_h u - u_h|_{1,h} \), which is equivalent to \( |I_h u - u_h|_1 \) by Lemma 4.1.

For comparison, Table 2 tabulates numerical results for the elliptic problem in Example 1 by using the finite-volume scheme based on the Simpson quadrature points. One can observe only a second-order convergence in \( |I_h u - u_h|_{1,h} \).

**Example 2** (A Parabolic Problem) The domain is \( \Omega = [0, \pi]^2 \), the permeability coefficient \( a(x, y) = (x + 1)^2 + y^2 \), the exact solution \( u(x, y, t) = e^{-0.1t} \sin(x) \sin(y) \) and the final time \( T = 0.2 \). The right-hand side \( f(x, y, t) \) is computed accordingly. This example was tested in Yang & Liu (2011).

In Table 3, we present numerical results obtained from using the Crank--Nicholson scheme for temporal discretization. We take a fixed but relatively small time step \( \Delta t = T/256 \) so that the spatial errors dominate. One can observe third-order convergence in \( h \) for \( \max_{0 \leq n \leq N} \| u^n - u_h^n \|_0 \) and
Table 1  Example 1: Numerical errors and convergence rates for the finite-volume scheme using Barlow points for the elliptic problem on quadrilateral meshes with $h = \pi / M$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$| u - u_h |_0$</th>
<th>Rate</th>
<th>$| u - u_h |_1$</th>
<th>Rate</th>
<th>$| I_h u - u_h |_{1,h}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.942E-4</td>
<td>—</td>
<td>7.704E-3</td>
<td>—</td>
<td>1.275E-3</td>
<td>—</td>
</tr>
<tr>
<td>16</td>
<td>5.231E-5</td>
<td>3.23</td>
<td>1.691E-3</td>
<td>2.18</td>
<td>7.223E-5</td>
<td>4.14</td>
</tr>
<tr>
<td>32</td>
<td>6.173E-6</td>
<td>3.08</td>
<td>4.051E-4</td>
<td>2.06</td>
<td>5.691E-6</td>
<td>3.66</td>
</tr>
<tr>
<td>64</td>
<td>7.595E-7</td>
<td>3.02</td>
<td>1.001E-4</td>
<td>2.01</td>
<td>5.948E-7</td>
<td>3.25</td>
</tr>
<tr>
<td>128</td>
<td>9.456E-8</td>
<td>3.00</td>
<td>2.495E-5</td>
<td>2.00</td>
<td>7.073E-8</td>
<td>3.07</td>
</tr>
</tbody>
</table>

Table 2  Example 1: Numerical errors and convergence rates for the finite-volume scheme using Simpson points for the elliptic problem on quadrilateral meshes with $h = \pi / M$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$| u - u_h |_0$</th>
<th>Rate</th>
<th>$| u - u_h |_1$</th>
<th>Rate</th>
<th>$| I_h u - u_h |_{1,h}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2.119E-3</td>
<td>—</td>
<td>8.272E-3</td>
<td>—</td>
<td>7.785E-3</td>
<td>—</td>
</tr>
<tr>
<td>16</td>
<td>4.537E-4</td>
<td>2.22</td>
<td>1.810E-3</td>
<td>2.19</td>
<td>1.618E-4</td>
<td>2.26</td>
</tr>
<tr>
<td>32</td>
<td>1.077E-4</td>
<td>2.07</td>
<td>4.329E-4</td>
<td>2.06</td>
<td>3.765E-4</td>
<td>2.10</td>
</tr>
<tr>
<td>64</td>
<td>2.650E-5</td>
<td>2.02</td>
<td>1.069E-4</td>
<td>2.01</td>
<td>9.203E-5</td>
<td>2.03</td>
</tr>
<tr>
<td>128</td>
<td>6.592E-6</td>
<td>2.00</td>
<td>2.664E-5</td>
<td>2.00</td>
<td>2.285E-5</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Table 3  Example 2: Numerical errors and convergence rates for the parabolic problem on quadrilateral meshes using the Crank–Nicholson scheme and Barlow points with fixed $\Delta t = T / 256$ and varying $h = \pi / M$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\max_{0 \leq n \leq N} | u^n - u_h^n |_0$</th>
<th>Rate</th>
<th>$\max_{0 \leq n \leq N} | u^n - u_h^n |_1$</th>
<th>Rate</th>
<th>$| I_h u^n - u_h^n |_E$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.999E-4</td>
<td>—</td>
<td>7.714E-3</td>
<td>—</td>
<td>5.525E-4</td>
<td>—</td>
</tr>
<tr>
<td>16</td>
<td>5.239E-5</td>
<td>3.25</td>
<td>1.691E-3</td>
<td>2.18</td>
<td>3.165E-5</td>
<td>4.12</td>
</tr>
<tr>
<td>32</td>
<td>6.175E-6</td>
<td>3.08</td>
<td>4.051E-4</td>
<td>2.06</td>
<td>2.531E-6</td>
<td>3.64</td>
</tr>
<tr>
<td>64</td>
<td>7.595E-7</td>
<td>3.02</td>
<td>1.001E-4</td>
<td>2.01</td>
<td>2.718E-7</td>
<td>3.21</td>
</tr>
<tr>
<td>128</td>
<td>1.010E-7</td>
<td>2.91</td>
<td>2.495E-4</td>
<td>2.00</td>
<td>4.853E-8</td>
<td>2.48</td>
</tr>
</tbody>
</table>

$max_{0 \leq n \leq N} h |u^n - u_h^n|_1$, as proved in Theorem 4.3. For $M = 128$, the $L^2$ and energy error convergence rates drop a bit. This is because the dominance of the spatial errors becomes weaker as $M$ increases.

In Table 4, we present numerical results obtained from using the third-order backward differentiation formula (BDF3) (Iserles, 1996) for temporal discretization. It is reflected in Table 4 that

$$
\| u^M - u_h^M \|_0 + h |u^M - u_h^M|_1 \lesssim h^3 + \Delta t^3, \quad 0 \leq M \leq N,
$$

when $\Delta t = o(h)$.

In Tables 3 and 4, we consider also a discrete temporal energy norm

$$
| I_h u - u_h|_E := \left( \Delta t \sum_{n=1}^{N} |I_h u^n - u_h^n|_{1,h}^2 \right)^{1/2}.
$$
Table 4: Example 2: Numerical errors and convergence rates for the parabolic problem on quadrilateral meshes using the BDF3 scheme and Barlow points with $h = \pi / M$, $\Delta t = (0.1/\pi)h$, $N = 2M$

| $M$  | $\max_{0 \leq n \leq N} \| u^n - u^n_h \|_0$ | Rate | $\max_{0 \leq n \leq N} | u^n - u^n_h |_1$ | Rate | $| l_h u - u_h |_E$ | Rate |
|------|-----------------------------------------------|------|---------------------------------------------|------|--------------------------------|------|
| 8    | 4.957E−4                                      | —    | 7.694E−3                                   | —    | 5.521E−4                      | —    |
| 16   | 5.232E−5                                      | 3.24 | 1.689E−3                                   | 2.18 | 3.156E−5                      | 4.12 |
| 32   | 6.171E−6                                      | 3.08 | 4.050E−4                                   | 2.06 | 2.506E−6                      | 3.65 |
| 64   | 7.594E−7                                      | 3.02 | 1.001E−3                                   | 2.01 | 2.640E−7                      | 3.24 |
| 128  | 9.458E−8                                      | 3.00 | 2.495E−5                                   | 2.00 | 3.256E−8                      | 3.01 |

Table 3 uses Crank–Nicholson for temporal discretization, whereas Table 4 uses BDF3. One can clearly observe better convergence rates in the energy norm from using the Barlow points than those from using Simpson points (see Yang & Liu, 2011, Table 2).

6. Concluding remarks

The usefulness of quadrilateral meshes for developing numerical methods for partial differential equations has been demonstrated in Arnold et al. (2002), Flemisch & Wohlmuth (2007), Li & Li (1999) and Schroll & Svensson (2006). Shape parameters for characterizing quality of quadrilateral meshes were discussed in Chou & He (2002) and Yang & Liu (2011). Mesh Assumption A and B are used for obtaining optimal error estimates, whereas Mesh Assumption C is used for deriving the superconvergence.

The new finite-volume schemes developed in this paper have optimal convergence rates, thanks to the Barlow points. Barlow points were originally discovered as optimal stress sampling points in finite bar elements (Barlow, 1976). It was then realized that Barlow points are the same as Gaussian points for linear and quadratic bar elements, but different for cubic bar elements (Júnior, 2008). The discrepancy has motivated alternative approaches, e.g., the variational approach, the best-fit method that leads to discovery of Prathap points and establishment of the relationships among Barlow points, Gaussian points and Prathap points for one-dimensional finite elements (Rajendran, 2009). Barlow points can be established for rectangular elements through tensor products and for quadrilateral elements via bilinear transformations. However, the study in Rajendran & Liew (2003) implies that the effectiveness of Barlow points, Gaussian points, and Prathap points for linear and quadratic finite (volume) elements on triangular meshes seems limited. On the other hand, construction of higher order Lagrange and Hermite-type finite-volume element methods on triangular elements are investigated in Chen et al. (2011) in a general framework. Development of higher order finite-volume methods remains an active and promising research front.

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