IMA Journal of Numerical Analysis (2013) **33**, 1342–1364
doi:10.1093/imanum/drs045
Advance Access publication on March 28, 2013

Quadratic finite-volume methods for elliptic and parabolic problems on quadrilateral meshes: optimal-order errors based on Barlow points

Min Yang*
Department of Mathematics, Yantai University, Yantai, Shandong, China
*Corresponding author: yang@ytu.edu.cn

Jiangguo Liu
Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874, USA
liu@math.colostate.edu

and

Yanping Lin
Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong and Department of Mathematical and Statistics Science, University of Alberta, Edmonton, AB, Canada T6G 2G1
malin@polyu.edu.hk yanlin@ualberta.ca

[Received on 10 December 2011; revised on 30 June 2012]

This paper presents quadratic finite-volume methods for elliptic and parabolic problems on quadrilateral meshes that use Barlow points (optimal stress points) for dual partitions. Introducing Barlow points into the finite-volume formulations results in better approximation properties at the cost of loss of symmetry. The novel 'symmetrization' technique adopted in this paper allows us to derive optimal-order error estimates in the H^1- and L^2-norms for elliptic problems and in the $L^\infty(H^1)$- and $L^\infty(L^2)$-norms for parabolic problems. Superconvergence of the difference between the gradients of the finite-volume solution and the interpolant can also be derived. Numerical results confirm the proved error estimates.

Keywords: Barlow points; error estimation; elliptic boundary value problems; finite-volume element methods; parabolic equations; quadrilateral meshes.

1. Introduction

Finite-volume methods have been widely used in scientific computing and engineering applications due to their local conservation properties and easy implementation. However, most finite-volume methods are low order. Development of higher order finite-volume methods has been an active research area, as reflected in Cai *et al.* (2003), Chen (2010), Chen *et al.* (2011), Gao & Wang (2010), Hackbusch (1989), Hyman *et al.* (1992), Plexousakis & Zouraris (2004), Wang & Gu (2010), Xu & Zou (2009), Yang (2006), Yang *et al.* (2009), Yang & Liu (2011), Yu & Li (2010, 2011) and the references therein.

This paper is a continuation of our efforts in Yang (2006), Yang *et al.* (2009), Yang & Liu (2011) on developing quadratic finite-volume methods for elliptic and parabolic problems. It was discussed in Yang & Liu (2011) that the quadratic finite-volume method based on the Simpson quadrature for parabolic problems on quadrilateral meshes has an optimal convergence rate in the $L^2(H^1)$-norm, but there are technical difficulties in deriving error estimates in the $L^\infty(H^1)$- and $L^\infty(L^2)$-norms that are

similar to those for the finite-element methods. The main reason for this deficiency is that we were unable to derive an optimal L^2-norm error estimate for the Ritz projection. Such deficiency exists on triangular and quadrilateral meshes, see Li *et al.* (2000), Liebau (1996), Xu & Zou (2009) and Yang (2006). In this paper, we utilize Barlow points (Barlow, 1976) for constructing dual volumes and developing quadratic finite-volume methods for elliptic and parabolic problems on quadrilateral meshes. Now we are able to derive optimal-order error estimates in the $L^\infty(H^1)$- and $L^\infty(L^2)$-norms. We have also superconvergence for the errors between the interpolation and the finite-volume solution. A novel contribution of this paper is the 'symmetrization' technique that enables us to overcome certain difficulties in the theoretical analysis.

Throughout the paper, we use the standard notations for the Sobolev spaces $W^{m,p}(\Omega)$ with the norm $\|\cdot\|_{m,p,\Omega}$ and the seminorm $|\cdot|_{m,p,\Omega}$. We also denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and skip the index $p=2$ and the domain Ω, when there is no ambiguity, that is, $\|u\|_{m,p}=\|u\|_{m,p,\Omega}$, $\|u\|_m=\|u\|_{m,2,\Omega}$. The same convention is adopted for the seminorms. We will also use $A\lesssim B$ and $B\gtrsim A$ to denote $A\leqslant CB$, where C is an absolute constant that may take different values in different appearances but is independent of spatial and temporal discretizations. Moreover, $A\sim B$ denotes that $A\lesssim B$ and $B\lesssim A$.

The rest of this paper is organized as follows. In Section 2, we discuss Barlow points, construction of dual volumes and mesh assumptions. Finite-volume methods for elliptic and parabolic problems on quadrilateral meshes are presented in Section 3. Section 4 is devoted to error estimation of the developed finite-volume schemes. Section 5 presents numerical results to illustrate the error estimates. Section 6 concludes the paper with some remarks.

2. Quadratic finite volumes and mesh assumptions

Let $\Omega_h=\{Q\}$ be a quadrilateral partition of Ω, where any two closed quadrilaterals share a common edge, vertex or nothing. Let $\hat{Q}=[-1,1]^2$ be the reference element in the $\hat{x}\hat{y}$-plane. For each element $Q\in\Omega_h$, there exists a bijective bilinear mapping $F_Q:\hat{Q}\longrightarrow Q$ satisfying

$$F_Q(\hat{P}_i)=P_i,\quad 1\leqslant i\leqslant 4.$$

Let $\mathscr{J}_{F_Q}$ be the Jacobian matrix of F_Q at $\hat{\mathbf{x}}$ and $J_{F_Q}=\det\mathscr{J}_{F_Q}$, and accordingly, $\mathscr{J}_{F_Q^{-1}}$ the Jacobian matrix of F_Q^{-1} at $\mathbf{x}$ and $J_{F_Q^{-1}}=\det\mathscr{J}_{F_Q^{-1}}$. Based on the partition Ω_h, we define S_h as the standard conforming finite-element space of piecewise affine biquadratic functions

$$S_h=\{v\in H_0^1(\Omega):\ v|_Q=\hat{v}\circ F_Q^{-1},\hat{v}|_{\hat{Q}}\text{ is biquadratic},\ \forall\, Q\in\Omega_h;\ v|_{\partial\Omega}=0\}. \tag{2.1}$$

Let $I_h:H_0^1(\Omega)\cap H^3(\Omega)\to S_h$ be the usual nodal interpolation operator. It is well known (see Brenner & Scott, 2008) that

$$\|u-I_hu\|_r\lesssim h^{3-r}\|u\|_3,\quad 0\leqslant r\leqslant 2. \tag{2.2}$$

In order to establish finite-volume element schemes, we introduce a dual partition Ω_h^*, whose elements are called control volumes. As shown in Fig. 1, each edge of $Q\in\Omega_h$ is partitioned into three segments so that the ratio of these segments is $1:\sqrt{3}+1:1$. We connect these partition points with line segments to the corresponding ones on the opposite edge. This way, each quadrilateral in Ω_h is divided into nine sub-quadrilaterals Q_z, $z\in Z_h(Q)$, where $Z_h(Q)$ is the set of the vertices, the midpoints of edges and the centre of Q. For each node $z\in Z_h=\bigcup_{Q\in\Omega_h}Z_h(Q)$, we associate a control volume V_z, which is the union of the subregions Q_z containing the node z. Therefore, we obtain a collection of

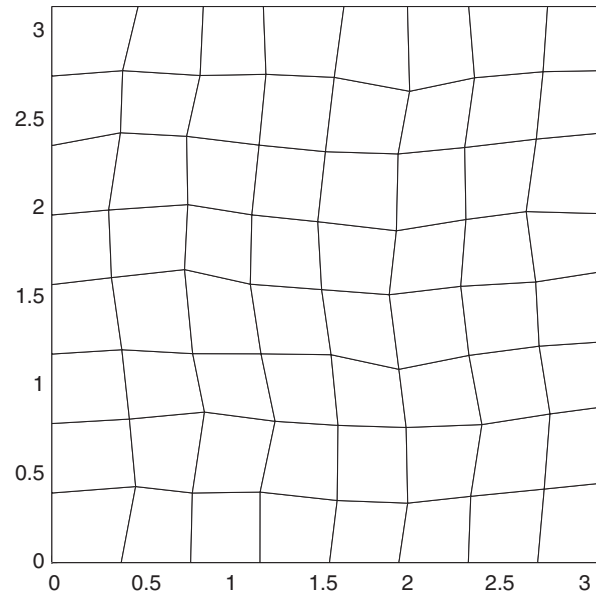

FIG. 1. An 8×8 quadrilateral mesh with random $\mathscr{O}(h)$-scale perturbations in both x, y-directions.

control volumes covering the domain Ω. This is the dual partition Ω_h^* of the primal partition Ω_h. We denote the set of interior nodes of Z_h by Z_h^0.

REMARK 2.1 The dual partition introduced here is different from the one defined in Yang & Liu (2011), where the sub-quadrilaterals Q_z are built by using the points related to the Simpson quadrature and the partition ratio 1:4:1. The control volumes in this paper are based on the Barlow points (the optimal stress points). The set of the Barlow points for one-dimensional Lagrange quadratic finite element is $N_2 = F\hat{N}_2$, where F is an invertible affine mapping from the reference element $[-1, 1]$ to a generic interval $[a, b]$, and

$$\hat{N}_2 = \left\{ -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\}.$$

Let $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and $L_i(x)$, $i = 1, 2, 3$ be the corresponding Lagrange quadratic basis functions. The derivative of the interpolant at the Barlow points satisfies (Barlow, 1976)

$$\left(x^3 - \sum_{i=1}^{3} x_i^3 L_i(x) \right)' \Bigg|_{x=\pm\sqrt{3}/3} = 0, \tag{2.3}$$

which means that the stresses are the most accurate at the Barlow points. The property (2.3) is critical for obtaining a superconvergence result for the term $a_h(u - I_h u, I_h^* v_h)$; see Lemma 4.3.

REMARK 2.2 Finite-volume methods based on the Barlow points (the optimal stress points) have been previously studied on one-dimensional (see, e.g., Gao & Wang, 2010) and two-dimensional rectangular meshes, see, e.g., Wang & Gu (2010) and Yu & Li (2010, 2011). But the extension of the methods to

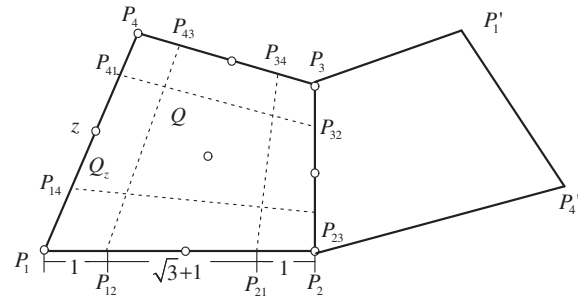

FIG. 2. Dual volumes and adjacent quadrilaterals.

more general quadrilateral meshes is not straightforward. The distortion of transformation might destroy the superconvergence property. A delicate analysis should be established to measure the effect of such distortion.

We make some assumptions on the quadrilateral mesh Ω_h as follows. For any $Q \in \Omega_h$, let h_Q be its diameter, h'_Q the smallest length of the edges and θ_Q any interior angle. We set $h = \max_{Q\in\Omega_h} h_Q$.

(1) *Mesh Assumption A*. The mesh $\Omega_h = \{Q\}$ is regular, that is, there exist two positive constants σ and γ such that

$$h_Q/h'_Q \leqslant \sigma, \quad |\cos\theta_Q| \leqslant \gamma < 1, \quad \forall\, Q \in \Omega_h. \tag{2.4}$$

(2) *Mesh Assumption B*. The quadrilateral mesh is 'asymptotically parallelogram'. Namely, for each element $Q \in \Omega_h$, one has

$$\|P_1 - P_2 + P_3 - P_4\| = \mathscr{O}(h^2), \tag{2.5}$$

where $P_i, i = 1, 2, 3, 4$ are the four vertices of Q.

(3) *Mesh Assumption C*. Any two adjacent quadrilaterals (see Fig. 2) form an h^2-parallelogram in the sense that

$$\|P_4 + P'_1 - 2P_3\| = \mathscr{O}(h^2). \tag{2.6}$$

These assumptions have been respectively adopted in Arnold *et al.* (2002), Chou & He (2002), Ewing *et al.* (1999) and Yang & Liu (2011).

3. Finite-volume schemes for elliptic and parabolic problems

We first consider a model elliptic boundary value problem

$$\begin{cases} \nabla\cdot(-a(\mathbf{x})\nabla u) = f(\mathbf{x}), & \mathbf{x}\in\Omega, \\ u = 0, & \mathbf{x}\in\partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\mathbf{x} = (x, y)$. It is assumed that $f(\mathbf{x}) \in L^2(\Omega)$ and $a(\mathbf{x})$ is Lipschitz continuous and bounded almost everywhere with positive lower and upper bounds a_* and a^*.

Given a node $z \in Z_h^0$, integrating the first equation in (3.1) over the control volume V_z and applying the Green's formula, we obtain

$$-\int_{\partial V_z} a\nabla u \cdot \mathbf{n}\,\mathrm{d}s = \int_{V_z} f\,\mathrm{d}\mathbf{x}, \tag{3.2}$$

where $\mathbf{n}$ denotes the unit outer normal on ∂V_z.

We further introduce a transfer operator $I_h^* : S_h \to S_h^*$ from the trial space to the test space such that

$$I_h^* v = \sum_{z \in Z_h^0} v(z)\Psi_z, \tag{3.3}$$

where

$$S_h^* = \{v \in L^2(\Omega) : v|_{V_z} \text{ is constant}, \ \forall z \in Z_h^0; \ v|_{V_z} = 0, \ \forall z \in \partial\Omega\}, \tag{3.4}$$

and Ψ_z is the characteristic function of the control volume V_z. Then we multiply (3.2) by $v_h(z)$ and sum over all $z \in Z_h^0$ to obtain an integral conservation form for elliptic problem

$$a_h(u, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in S_h, \tag{3.5}$$

where the bilinear form $a_h(\cdot, I_h^* \cdot)$ is defined as: for any $u \in H_0^1(\Omega)$, $v_h \in S_h$,

$$a_h(u, I_h^* v_h) = -\sum_{z \in Z_h^0} v_h(z) \int_{\partial V_z} a\nabla u \cdot \mathbf{n}\,\mathrm{d}s. \tag{3.6}$$

A finite-volume scheme for elliptic problem (3.1) is formulated as: Seek $u_h \in S_h$ such that

$$a_h(u_h, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in S_h. \tag{3.7}$$

A model parabolic initial boundary value problem can be formulated as

$$\begin{cases} u_t - \nabla \cdot (a(\mathbf{x})\nabla u) = f(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{3.8}$$

where $f(\mathbf{x}, t) \in L^2(\Omega)$ for $t \in [0, T]$ and other terms follow the same assumptions for the elliptic problem.

A semidiscrete finite-volume scheme for (3.8) is accordingly defined as: Seek $u_h(t) \in S_h$ for $t \in (0, T]$ such that for any $v_h \in S_h$,

$$(u_{h,t}, I_h^* v_h) + a_h(u_h, I_h^* v_h) = (f, I_h^* v_h) \tag{3.9}$$

with an initial approximation $u_h(0)$ given by $u_h(0) = R_h u_0$, where $R_h : H_0^1(\Omega) \cap H^3(\Omega) \to S_h$ is the elliptic (Ritz) projection defined by

$$a_h(R_h u, I_h^* v_h) = a_h(u, I_h^* v_h), \quad \forall v_h \in S_h. \tag{3.10}$$

REMARK 3.1 Let $\{\Phi_z : z \in Z_h^0\}$ and $\{\Psi_z : z \in Z_h^0\}$ be the standard basis functions of S_h and S_h^*, respectively. Then scheme (3.9) can be rewritten as a system of ordinary differential equations

$$\mathscr{M}\alpha'(t) + \mathscr{S}\alpha(t) = \tilde{f}(t), \quad 0 \leqslant t \leqslant T; \quad \alpha(0) = \beta, \tag{3.11}$$

where $\mathscr{M} = ((\Phi_z, \Psi_w))_{zw}$ and $\mathscr{S} = (a_h(\Phi_z, \Psi_w))_{zw}$ are the mass and stiffness matrices, respectively, and $\alpha(t)$ and β are vectors of the nodal values of $u_h(t)$ and $R_h u_0$, respectively. From Lemmas 4.1 and 4.4, we know that both $\mathscr{M}$ and $\mathscr{S}$ are invertible. This implies that there exists a unique solution $u_h(\cdot, t)$ on $\Omega \times [0, T]$.

Let N be a positive integer. For simplicity of presentation, we consider a uniform time step $\Delta t = T/N$ and set $t_n = n\Delta t$ $(0 \leqslant n \leqslant N)$. For $n \geqslant 1$, let

$$\bar{\partial} u_h^n = \frac{u_h^n - u_h^{n-1}}{\Delta t}, \quad u_h^{n,1/2} = \frac{u_h^n + u_h^{n-1}}{2}.$$

A Crank–Nicolson fully discrete (Thomée, 2006) *finite-volume scheme for* (3.8) seeks $u_h^n \in S_h$ such that for any $v_h \in S_h$,

$$(\bar{\partial} u_h^n, I_h^* v_h) + a_h(u_h^{n,1/2}, I_h^* v_h) = (f^{n,1/2}, I_h^* v_h), \quad n \geqslant 1, \tag{3.12}$$

with an initial approximation given by $u_h^0 = R_h u_0$.

REMARK 3.2 The scheme (3.12) takes a form as follows

$$(\mathscr{M} + \tfrac{1}{2}\Delta t\mathscr{S})\alpha^n = (\mathscr{M} - \tfrac{1}{2}\Delta t\mathscr{S})\alpha^{n-1} + \Delta t\tilde{f}^{n,1/2},$$

Generally, $\mathscr{M}$ and $\mathscr{S}$ are not symmetric. But by Lemmas 4.1 and 4.4, we know that both $\mathscr{M} + \mathscr{M}^{\mathrm{T}}$ and $\mathscr{S} + \mathscr{S}^{\mathrm{T}}$ are positive definite. So, for any nonzero vector $\mathbf{x}$,

$$\mathbf{x}^{\mathrm{T}}(\mathscr{M} + \tfrac{1}{2}\Delta t\mathscr{S})\mathbf{x} = \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}(\mathscr{M} + \mathscr{M}^{\mathrm{T}})\mathbf{x} + \tfrac{1}{4}\Delta t\mathbf{x}^{\mathrm{T}}(\mathscr{S} + \mathscr{S}^{\mathrm{T}})\mathbf{x} > 0,$$

which means $\mathscr{M} + \frac{1}{2}\Delta t\mathscr{S}$ is invertible. Therefore, (3.12) can be solved uniquely at each time step.

4. Error estimates

For simplicity of presentation, we assume that the coefficient $a(x) \equiv 1$ in this section. If $a(x)$ is non-constant, we can have a perturbation argument by taking the piecewise constant approximation in each element Q. Then all results will still hold for h small enough; see Yang & Liu (2011) for details.

4.1 *Basic approximation properties*

It is assumed that there exist a pair of integers n_x, n_y such that the cardinality of Ω_h is equal to $n_x n_y$, and we can assign each $Q \in \Omega_h$ a pair of integers (i, j), where $0 \leqslant i \leqslant n_x - 1$, $0 \leqslant j \leqslant n_y - 1$. Thus we label Q by subscripts (i, j) and denote its vertices by $\mathbf{x}_{i,j}, \mathbf{x}_{i+1,j}, \mathbf{x}_{i+1,j+1}, \mathbf{x}_{i,j+1}$, corresponding to P_1, P_2, P_3, P_4 in Fig. 2. Let ν_i, $\nu_j = 0$ or $\frac{1}{2}$. Then the midpoints of the edges of Q are denoted by $\mathbf{x}_{i+\nu_i, j+\nu_j}$, where $\nu_i + \nu_j = \frac{1}{2}$, and the centre of Q denoted by $\mathbf{x}_{i+1/2, j+1/2}$.

Similar to Yang & Liu (2011), we now define some discrete norms on S_h. For any $u_h \in S_h$,

$$|\|u_h\||_0^2 = (u_h, I_h^* u_h), \tag{4.1}$$

$$|u_h|_{1,h}^2 = \sum_{Q\in\Omega_h} |u_h|_{1,h,Q}^2 = \sum_{Q\in\Omega_h} \left[\sum_{\nu_i=1/2,1} \sum_{\nu_j=0,1/2,1} (\delta_x u_h(\mathbf{x}_{i+\nu_i,j+\nu_j}))^2 \right.$$
$$\left. + \sum_{\nu_i=0,1/2,1} \sum_{\nu_j=1/2,1} (\delta_y u_h(\mathbf{x}_{i+\nu_i,j+\nu_j}))^2 \right], \tag{4.2}$$

where

$$\delta_x u_h(\mathbf{x}_{i+\nu_i,j+\nu_j}) = u_h(\mathbf{x}_{i+\nu_i,j+\nu_j}) - u_h(\mathbf{x}_{i+\nu_i-\frac{1}{2},j+\nu_j}),$$
$$\delta_y u_h(\mathbf{x}_{i+\nu_i,j+\nu_j}) = u_h(\mathbf{x}_{i+\nu_i,j+\nu_j}) - u_h(\mathbf{x}_{i+\nu_i,j+\nu_j-\frac{1}{2}}).$$

The following lemma reveals that the discrete norms are equivalent to the continuous norms.

LEMMA 4.1 Assume that Ω_h satisfies Mesh Assumption A. For any $u_h \in S_h$, we have

$$\|\,|u_h|\,\|_0 \sim \|u_h\|_0, \tag{4.3}$$
$$|u_h|_{1,h} \sim |u_h|_1. \tag{4.4}$$

Proof. Since the mesh is regular, we have

$$\|\hat{u}_h\|_{0,\hat{Q}} \leqslant \|J_{F_Q^{-1}}\|_{\infty,Q}^{1/2} \|u_h\|_{0,Q}, \quad \|u_h\|_{0,Q} \leqslant \|J_{F_Q}\|_{\infty,\hat{Q}}^{1/2} \|\hat{u}_h\|_{0,\hat{Q}},$$

where

$$\|J_{F_Q^{-1}}\|_{\infty,Q} \lesssim h_Q^{-2}, \quad \|J_{F_Q}\|_{\infty,\hat{Q}} \lesssim h_Q^2.$$

Therefore,

$$h_Q \|\hat{u}_h\|_{0,\hat{Q}} \lesssim \|u_h\|_{0,Q} \lesssim h_Q \|\hat{u}_h\|_{0,\hat{Q}}. \tag{4.5}$$

For $\int_Q u_h I_h^* u_h \,\mathrm{d}\mathbf{x}$, we have a similar estimate as follows

$$h_Q^2 \int_{\hat{Q}} \hat{u}_h^2 \widehat{I_h^* u_h} \,\mathrm{d}\hat{\mathbf{x}} \lesssim \int_Q u_h I_h^* u_h \,\mathrm{d}\mathbf{x} \lesssim h_Q^2 \int_{\hat{Q}} \hat{u}_h^2 \widehat{I_h^* u_h} \,\mathrm{d}\hat{\mathbf{x}}. \tag{4.6}$$

Let

$$\varphi_1(x) = \tfrac{1}{2}x(x-1), \quad \varphi_2(x) = (1-x)(x+1), \quad \varphi_3(x) = \tfrac{1}{2}x(x+1)$$

be the local quadratic basis on $[-1, 1]$. Let $\psi_i(x), 1 \leqslant i \leqslant 3$ be the characteristic functions associated with the partitions $[-1, -\sqrt{3}/3]$, $[-\sqrt{3}/3, \sqrt{3}/3]$ and $[\sqrt{3}/3, 1]$, respectively. Then using the standard tensor-product basis and the resulting interpolation form of $\hat{u}_h$ on the reference element, we obtain

immediately

$$\|\hat{u}_h\|_{0,\hat{Q}}^2 = \mathbf{u}_Q(\mathscr{G}_1 \otimes \mathscr{G}_1)\mathbf{u}_Q^{\mathrm{T}}, \quad \int_{\hat{Q}} \hat{u}_h \widehat{I_h^* u_h}\,\mathrm{d}\hat{\mathbf{x}} = \mathbf{u}_Q(\mathscr{G}_2 \otimes \mathscr{G}_2)\mathbf{u}_Q^{\mathrm{T}} = \frac{1}{2}\mathbf{u}_Q(\mathscr{G}_2 \otimes \mathscr{G}_2 + \mathscr{G}_2^{\mathrm{T}} \otimes \mathscr{G}_2^{\mathrm{T}})\mathbf{u}_Q^{\mathrm{T}},$$

where $\mathbf{u}_Q \in \mathbb{R}^9$ is a vector consisting of the nodal values of u_h on Q and

$$\mathscr{G}_1 = \left(\int_{-1}^{1} \varphi_i\varphi_j\,\mathrm{d}x\right)_{i,j=1}^{3} = \frac{1}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$\mathscr{G}_2 = \left(\int_{-1}^{1} \psi_i\varphi_j\,\mathrm{d}x\right)_{i,j=1}^{3} = \frac{1}{54}\begin{bmatrix} 18-\sqrt{3} & 36-16\sqrt{3} & -\sqrt{3} \\ 2\sqrt{3} & 32\sqrt{3} & 2\sqrt{3} \\ -\sqrt{3} & 36-16\sqrt{3} & 18-\sqrt{3} \end{bmatrix}.$$

It is easy to verify that $\mathscr{G}_1$ and $\mathscr{G}_2 \otimes \mathscr{G}_2 + \mathscr{G}_2^{\mathrm{T}} \otimes \mathscr{G}_2^{\mathrm{T}}$ are symmetric and positive definite. Thus $\|\hat{u}_h\|_{0,\hat{Q}}^2$ and $\int_{\hat{Q}} \hat{u}_h \widehat{I_h^* u_h}\,\mathrm{d}\hat{\mathbf{x}}$ are equivalent. Applying (4.5) and (4.6) and summing the result over Ω_h yield estimate (4.3). Estimate (4.4) was proved in Yang (2006). □

We shall often use the Bramble–Hilbert lemma (Brenner & Scott, 2008) as recapped below.

Lemma 4.2 (Bramble–Hilbert lemma) Let $L(u)$ be a continuous linear functional on $W^{m,p}(\Omega)$ and $W^{m,p}(\Omega)'$ be its dual norm. If $L(u)$ vanishes for all $u \in \mathbb{P}^{m-1}$, then there exists a constant $C = C(\Omega)$ such that

$$|L(u)| \leqslant C|L|_{W^{m,p}(\Omega)'}|u|_{W^{m,p}(\Omega)}. \tag{4.7}$$

4.2 *Error estimates for the elliptic problem*

The bilinear form $a_h(u, I_h^* v_h)$ can be rewritten as

$$\begin{aligned} a_h(u, I_h^* v_h) &= \sum_{Q\in\Omega_h} a_{Q,h}(u, I_h^* v_h) = \sum_{Q\in\Omega_h}\left(-\sum_{z\in Z_h\cap Q} v_h(z)\int_{\partial V_z\cap Q} \nabla u\cdot\mathbf{n}\,\mathrm{d}s\right) \\ &= \sum_{Q\in\Omega_h}\left(\sum_{z_1,z_2\in Z_h\cap Q}(v_h(z_1)-v_h(z_2))\int_{\partial V_{z_1}\cap\partial V_{z_2}} \nabla u\cdot\mathbf{n}\,\mathrm{d}s\right), \end{aligned} \tag{4.8}$$

where z_1, z_2 are chosen in Q with no repetition. Applying the affine transformation F_Q, we have

$$\int_{\partial V_{z_1}\cap\partial V_{z_2}} \nabla u\cdot\mathbf{n}\,\mathrm{d}s = -\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}} (\hat{u}_{\hat{x}}b_{11} + \hat{u}_{\hat{y}}b_{12})\,\mathrm{d}\hat{y} + \int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}} (\hat{u}_{\hat{x}}b_{21} + \hat{u}_{\hat{y}}b_{22})\,\mathrm{d}\hat{x}, \tag{4.9}$$

where

$$b_{11} = J_{F_Q}^{-1}\left\{\left(\frac{\partial x}{\partial \hat{y}}\right)^2 + \left(\frac{\partial y}{\partial \hat{y}}\right)^2\right\},$$

$$b_{12} = b_{21} = -J_{F_Q}^{-1}\left\{\frac{\partial x}{\partial \hat{x}}\frac{\partial x}{\partial \hat{y}} + \frac{\partial y}{\partial \hat{x}}\frac{\partial y}{\partial \hat{y}}\right\},$$

$$b_{22} = J_{F_Q}^{-1}\left\{\left(\frac{\partial x}{\partial \hat{x}}\right)^2 + \left(\frac{\partial y}{\partial \hat{x}}\right)^2\right\}.$$

Differentiating the identity $J_{F_Q}J_{F_Q^{-1}} = 1$ and applying math induction, we have $D^\alpha J_{F_Q^{-1}} = \mathscr{O}(h_Q^{-2+|\alpha|})$. Then the estimate (Zlámal, 1978)

$$|D^\alpha b_{ij}| \lesssim h_Q^{|\alpha|}, \quad \alpha \geqslant 0, \quad i,j = 1,2 \tag{4.10}$$

follows from the Leibniz rule.

Next we prove a superapproximation result for $a_h(u - I_h u, I_h^* v_h)$.

LEMMA 4.3 Assume that Ω_h satisfies Mesh Assumptions A, B and C. If $u \in H^4(\Omega)$ and $v_h \in S_h$, then

$$|a_h(u - I_h u, I_h^* v_h)| \lesssim h^3 \|u\|_4 |v_h|_1. \tag{4.11}$$

Proof. Let $w = u - I_h u$. Using (4.8) and (4.9) gives

$$a_h(w, I_h^* v_h) = \sum_{Q \in \Omega_h} a_{Q,h}(w, I_h^* v_h)$$

with

$$\begin{aligned} a_{Q,h}(w, I_h^* v_h) = &- \sum_{\hat{z}_1, \hat{z}_2 \in \hat{Z}_h \cap \hat{Q}} (\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)) \int_{\partial V_{\hat{z}_1} \cap \partial V_{\hat{z}_2}} (\hat{w}_{\hat{x}} b_{11} + \hat{w}_{\hat{y}} b_{12})\, \mathrm{d}\hat{y} \\ &+ \sum_{\hat{z}_1, \hat{z}_2 \in \hat{Z}_h \cap \hat{Q}} (\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)) \int_{\partial V_{\hat{z}_1} \cap \partial V_{\hat{z}_2}} (\hat{w}_{\hat{x}} b_{21} + \hat{w}_{\hat{y}} b_{22})\, \mathrm{d}\hat{x}. \end{aligned} \tag{4.12}$$

We only need to estimate the first term which includes the line integral along the $\hat{y}$ direction. The integral along the $\hat{x}$ direction can be estimated similarly.

Note that $\hat{x} = \pm\sqrt{3}/3$ in the first integral. By estimate (2.3) and the definition of the interpolation I_h, we have $\hat{w}_{\hat{x}} = 0$ for $\hat{u} = \hat{x}^i \hat{y}^j, 0 \leqslant i + j \leqslant 3$. Hence, the Bramble–Hilbert lemma and (4.10) yield

$$\begin{aligned} &- \sum_{\hat{z}_1, \hat{z}_2 \in \hat{Z}_h \cap \hat{Q}} (\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)) \int_{\partial V_{\hat{z}_1} \cap \partial V_{\hat{z}_2}} \hat{w}_{\hat{x}} b_{11}\, \mathrm{d}\hat{y} \lesssim |\hat{u}|_{4,\hat{Q}} \sum_{\hat{z}_1, \hat{z}_2 \in \hat{Z}_h \cap \hat{Q}} |\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)| \\ &\lesssim \sum_{z_1, z_2 \in Z_h \cap Q} h_Q^3 \|u\|_{4,Q} |v_h(z_1) - v_h(z_2)| \lesssim h^3 \|u\|_{4,Q} |v_h|_{1,h,Q}. \end{aligned} \tag{4.13}$$

The estimation for

$$-\sum_{\hat{z}_1,\hat{z}_2\in\hat{Z}_h\cap\hat{Q}}(\hat{v}_h(\hat{z}_1)-\hat{v}_h(\hat{z}_2))\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}\hat{w}_{\hat{y}}b_{12}\,\mathrm{d}\hat{y}$$

is technically involved. A starting point is to use (2.3) to get a superconvergence. But note that $\hat{y}\in[-1,1]$ and $\hat{w}_{\hat{y}}$ vanishes only for $\hat{u}=\hat{x}^i\hat{y}^j, 0\leqslant i,j\leqslant 2$. So, we shall construct a quadrature formula, in which $\hat{y}$ is fixed at $\pm\sqrt{3}/3$, to approximate the integral. Then applying Mesh Assumption C, we find that the lower order parts of the sum of the truncation terms on all elements can be cancelled on the element edges.

For each element Q, let $\bar{b}_{12}$ be the average value of b_{12}. By (4.10), we have $b_{12}=\bar{b}_{12}+\mathscr{O}(h_Q)$. Then

$$\begin{aligned}&-\sum_{\hat{z}_1,\hat{z}_2\in\hat{Z}_h\cap\hat{Q}}(\hat{v}_h(\hat{z}_1)-\hat{v}_h(\hat{z}_2))\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}\hat{w}_{\hat{y}}b_{12}\,\mathrm{d}\hat{y}\\&\leqslant-\bar{b}_{12}\sum_{\hat{z}_1,\hat{z}_2\in\hat{Z}_h\cap\hat{Q}}(\hat{v}_h(\hat{z}_1)-\hat{v}_h(\hat{z}_2))\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}\hat{w}_{\hat{y}}\,\mathrm{d}\hat{y}+Ch^3\|u\|_{3,Q}|v_h|_{1,h,Q}.\end{aligned}\tag{4.14}$$

We introduce a quadrature formula to approximate $\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}\hat{w}_{\hat{y}}\,\mathrm{d}\hat{y}$. For any function $f(\hat{y})$,

$$\int_{-1}^{-\sqrt{3}/3}f\,\mathrm{d}\hat{y}\approx\int_{-1}^{-\sqrt{3}/3}f\left(-\frac{\sqrt{3}}{3}\right)\mathrm{d}\hat{y},\quad\int_{\sqrt{3}/3}^{1}f\,\mathrm{d}\hat{y}\approx\int_{\sqrt{3}/3}^{1}f\left(\frac{\sqrt{3}}{3}\right)\mathrm{d}\hat{y}$$

and

$$\int_{-\sqrt{3}/3}^{\sqrt{3}/3}f\,\mathrm{d}\hat{y}\approx\frac{1}{2}\int_{-\sqrt{3}/3}^{\sqrt{3}/3}\left(f\left(-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)\right)\mathrm{d}\hat{y}.$$

Denoting by $\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}\hat{w}_{\hat{y}}^{\mathrm{app}}\,\mathrm{d}\hat{y}$ the approximation of $\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}\hat{w}_{\hat{y}}\,\mathrm{d}\hat{y}$. Noting that $\hat{y}=\pm\sqrt{3}/3$ in $\hat{w}_{\hat{y}}^{\mathrm{app}}$, we have

$$-\bar{b}_{12}\sum_{\hat{z}_1,\hat{z}_2\in\hat{Z}_h\cap\hat{Q}}(\hat{v}_h(\hat{z}_1)-\hat{v}_h(\hat{z}_2))\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}\hat{w}_{\hat{y}}d\hat{y}\leqslant\bar{b}_{12}R_1+Ch^3\|u\|_{4,Q}|v_h|_{1,h,Q},\tag{4.15}$$

where

$$R_1=\sum_{\hat{z}_1,\hat{z}_2\in\hat{Z}_h\cap\hat{Q}}(\hat{v}_h(\hat{z}_1)-\hat{v}_h(\hat{z}_2))\int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}}(\hat{w}_{\hat{y}}^{\mathrm{app}}-\hat{w}_{\hat{y}})\,\mathrm{d}\hat{y}.$$

To estimate term R_1, we construct an auxiliary functional

$$F_1=\frac{\sqrt{3}}{18}\left\{\int_{-1}^{1}\frac{\partial^2\hat{w}(1,\hat{y})}{\partial\hat{y}^2}\frac{\partial\hat{v}_h(1,\hat{y})}{\partial\hat{y}}\,\mathrm{d}\hat{y}-\int_{-1}^{1}\frac{\partial^2\hat{w}(-1,\hat{y})}{\partial\hat{y}^2}\frac{\partial\hat{v}_h(-1,\hat{y})}{\partial\hat{y}}\,\mathrm{d}\hat{y}\right\}.$$

Set $L(\hat{u})=R_1+F_1$. If $\hat{u}=\hat{x}^i\hat{y}^j, 0\leqslant i,j\leqslant 2$, then $\hat{w}=0$. If $\hat{u}=\hat{x}^3$, then $\hat{w}_y=0$. If $\hat{u}=\hat{y}^3$, a direct calculation produces

$$R_1|_{\hat{u}=\hat{y}^3}=-F_1|_{\hat{u}=\hat{y}^3}=-\frac{4\sqrt{3}}{9}\frac{\partial^3\hat{v}_h}{\partial\hat{x}\partial\hat{y}^2}(0,0).$$

Thus $L(\hat{u})$ vanishes for $\hat{u} = \hat{x}^i\hat{y}^j$, $0 \leqslant i + j \leqslant 3$. Note that $L(\hat{u})$ is a linear functional of $\hat{u}$. By the Bramble–Hilbert lemma and the norm equivalence in the finite-dimensional spaces, we have

$$\begin{aligned} L(\hat{u}) &\lesssim \|\hat{u}\|_{4,\hat{Q}}(|\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)| + \|\hat{v}_h\|_{H^1(\partial\hat{Q})}) \\ &\lesssim \|\hat{u}\|_{4,\hat{Q}}(|\hat{v}_h|_{1,h,\hat{Q}} + |\hat{v}_h|_{1,\hat{Q})}) \lesssim h^3\|u\|_{4,Q}|v_h|_{1,Q}. \end{aligned} \tag{4.16}$$

Since $R_1 = L(\hat{u}) - F_1$, combining (4.14) to (4.16) together yields

$$-\sum_{\hat{z}_1,\hat{z}_2\in\hat{Z}_h\cap\hat{Q}} (\hat{v}_h(\hat{z}_1) - \hat{v}_h(\hat{z}_2)) \int_{\partial V_{\hat{z}_1}\cap\partial V_{\hat{z}_2}} \hat{w}_{\hat{y}} b_{12}\,\mathrm{d}\hat{y} \lesssim h^3\|u\|_{4,Q}|v_h|_{1,Q} - \bar{b}_{12}F_1. \tag{4.17}$$

The estimation for the second integral in (4.12) is similar. Hence,

$$a_{Q,h}(w, I_h^* v_h) \lesssim h^3\|u\|_{4,Q}(|v_h|_{1,h,Q} + |v_h|_{1,Q}) - \bar{b}_{12}F_1 - \bar{b}_{21}F_2,$$

where F_2 is a functional similar to F_1, with $\hat{y}$ being replaced by $\hat{x}$. Therefore,

$$|a_h(w, I_h^* v_h)| \lesssim h^3\|u\|_4(|v_h|_{1,h} + |v_h|_1) + \left|\sum_{Q\in\Omega_h} \bar{b}_{12}F_1\right| + \left|\sum_{Q\in\Omega_h} \bar{b}_{21}F_2\right|. \tag{4.18}$$

The sum $\sum_{Q\in\Omega_h} \bar{b}_{ij}F_i(\hat{u})$ involves integrals on the edges of Q. On $\partial Q \cap \partial\Omega$, the corresponding integrals vanish because $v_h|_{\partial\Omega} = 0$. On an interior edge $\partial Q \cap \partial Q'$, the integrals from Q and Q' are taken for the same integrand but in opposite directions. Thus we can regroup the sum by edges to obtain the following terms on the interior edges

$$\frac{\sqrt{3}}{18}(\bar{b}_{12} - \bar{b}'_{12}) \int_{-1}^{1} \frac{\partial^2\hat{w}(1,\hat{y})}{\partial\hat{y}^2}\frac{\partial\hat{v}_h(1,\hat{y})}{\partial\hat{y}}\,\mathrm{d}\hat{y}, \quad \frac{\sqrt{3}}{18}(\bar{b}_{21} - \bar{b}'_{21}) \int_{-1}^{1} \frac{\partial^2\hat{w}(\hat{y},1)}{\partial\hat{x}^2}\frac{\partial\hat{v}_h(\hat{x},1)}{\partial\hat{x}}\,\mathrm{d}\hat{x}.$$

Under our mesh assumptions, $|\bar{b}_{ij} - \bar{b}'_{ij}|$ is $\mathscr{O}(h)$ (see, e.g., Zlámal, 1978). Therefore, it follows from the Bramble–Hilbert lemma that

$$\begin{aligned} &\left|\sum_{Q\in\Omega_h} \bar{b}_{12}F_1(\hat{u})\right| + \left|\sum_{Q\in\Omega_h} \bar{b}_{21}F_2(\hat{u})\right| \\ &\lesssim h\sum_{Q\in\Omega_h}\left(\left|\int_{-1}^{1} \frac{\partial^2\hat{w}(1,\hat{y})}{\partial\hat{y}^2}\frac{\partial\hat{v}_h(1,\hat{y})}{\partial\hat{y}}\,\mathrm{d}\hat{y}\right| + \left|\int_{-1}^{1} \frac{\partial^2\hat{w}(\hat{x},1)}{\partial\hat{x}^2}\frac{\partial\hat{v}_h(\hat{x},1)}{\partial\hat{x}}\,\mathrm{d}\hat{x}\right|\right) \\ &\lesssim h\sum_{Q\in\Omega_h}\|\hat{u}\|_{3,\hat{Q}}|\hat{v}_h|_{1,\hat{Q}} \lesssim h^3\|u\|_3|v_h|_1. \end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19) together and using the norm equivalence give the desired result. □

LEMMA 4.4 Assume that Ω_h satisfies Mesh Assumptions A and B. Then

$$|a_h(u_h, I_h^* v_h)| \lesssim |u_h|_1|v_h|_1, \quad \forall\, u_h, v_h \in S_h, \tag{4.20}$$

$$a_h(u_h, I_h^* u_h) \gtrsim |u_h|_1^2, \quad \forall\, u_h \in S_h. \tag{4.21}$$

Proof. The continuity (4.20) has been proved in Yang & Liu (2011). The coercivity (4.21) holds when $|\cos\theta_Q| < 0.99$. The proof is to decompose the distortion of affine mappings into contraction and rotation and to employ partitioned matrices for size reduction, which is the same as that for Lemma 3.8 in Yang & Liu (2011) and is also similar to that for Lemma 4.5 in this paper. □

THEOREM 4.1 Let u be the solution of (3.1) and u_h the finite-volume solution of (3.7). Assume that $u \in H^4(\Omega) \cap H_0^1(\Omega)$. If Mesh Assumptions A, B and C are satisfied, then

$$\|u - u_h\|_0 + h|u - u_h|_1 + |I_h u - u_h|_1 \lesssim h^3 \|u\|_4. \tag{4.22}$$

Proof. We decompose the error as $u_h - u = \xi - \eta$, where $\xi = u_h - I_h u$ and $\eta = u - I_h u$. By (3.5), (3.7), and Lemma 4.4, we have

$$|\xi|_1^2 \lesssim a_h(\xi, I_h^* \xi) = a_h(\eta, I_h^* \xi).$$

It follows from Lemma 4.3 that

$$a_h(\eta, I_h^* \xi) \lesssim h^3 \|u\|_4 |\xi|_1.$$

Therefore,

$$|\xi|_1 \lesssim h^3 \|u\|_4.$$

The approximation property (2.2) and a triangle inequality lead to

$$|u - u_h|_1 \leqslant |\xi|_1 + |\eta|_1 \lesssim h^2 \|u\|_4$$

and

$$\|u - u_h\|_0 \leqslant \|\xi\|_0 + \|\eta\|_0 \lesssim |\xi|_1 + \|\eta\|_0 \lesssim h^3 \|u\|_4,$$

which complete the proof. □

REMARK 4.1 We have obtained a superapproximation result for $|I_h u - u_h|_1$ in Theorem 4.1. This superconvergence not only helps us to derive optimal-order error estimates in the L^2-norm, but can also be used for developing *a posteriori* error estimators or postprocessing algorithms, which will be pursued in our future work. The superconvergence is based on the Barlow points and Mesh Assumption A, B and C. If only Mesh Assumptions A and B are considered, then the optimal order L^2 error estimate can also be obtained by using a duality argument and comparison with the finite-element bilinear forms.

4.3 *Error estimates for the parabolic problem*

According to Lemma 4.1, utilizing the Barlow points results in the term

$$\int_{\hat{Q}} \hat{u}_h \widehat{I_h^* v_h} \, \mathrm{d}\hat{\mathbf{x}} = \mathbf{v}_Q^{\mathrm{T}} \mathscr{G}_2 \otimes \mathscr{G}_2 \mathbf{u}_Q \tag{4.23}$$

being nonsymmetric, since $\mathscr{G}_2$ is nonsymmetric. We intend to 'symmetrize' the above bilinear form on the reference element.

LEMMA 4.5 For any $u_h, v_h \in S_h$, there exist $\tilde{u}_h, \tilde{v}_h \in S_h$ such that

$$\int_{\hat{Q}} \hat{u}_h \widehat{I_h^* \tilde{v}_h}\, \mathrm{d}\hat{\mathbf{x}} = \int_{\hat{Q}} \hat{v}_h \widehat{I_h^* \tilde{u}_h}\, \mathrm{d}\hat{\mathbf{x}}, \tag{4.24}$$

$$(u_h, I_h^* \tilde{u}_h)^{1/2} \sim \|u_h\|_0, \tag{4.25}$$

$$|\tilde{u}_h|_{1,h} \sim |u_h|_{1,h}, \tag{4.26}$$

$$a_h(u_h, I_h^* \tilde{u}_h) \gtrsim |u_h|_1^2. \tag{4.27}$$

Proof. The proof consists of three major steps.

The first step is to construct $\tilde{u}_h$ for any $u_h \in S_h$. Assume that $\tilde{u}_h$ has the form $\tilde{\mathbf{u}}_Q^{\mathrm{T}} = \mathbf{u}_Q^{\mathrm{T}} \mathscr{D} \otimes \mathscr{D}$. Utilizing a computer algebra system, e.g., Matlab, we obtain (generally not unique though)

$$\mathscr{D} = \frac{1}{6} \begin{bmatrix} 6 & 3-2\sqrt{3} & 0 \\ 0 & 4\sqrt{3} & 0 \\ 0 & 3-2\sqrt{3} & 6 \end{bmatrix}.$$

The second step is to verify (4.24–4.26). By the proof of Lemma 3.1, we have

$$\begin{aligned} \int_{\hat{Q}} \hat{u}_h \widehat{I_h^* \tilde{v}_h}\, \mathrm{d}\hat{\mathbf{x}} &= \sum_{Q \in \Omega_h} \tilde{\mathbf{v}}_Q^{\mathrm{T}} \mathscr{G}_2 \otimes \mathscr{G}_2 \mathbf{u}_Q \\ &= \sum_{Q \in \Omega_h} (\mathbf{v}_Q^{\mathrm{T}} \mathscr{D} \otimes \mathscr{D}) \mathscr{G}_2 \otimes \mathscr{G}_2 \mathbf{u}_Q = \sum_{Q \in \Omega_h} \mathbf{v}_Q^{\mathrm{T}} (\mathscr{D}\mathscr{G}_2) \otimes (\mathscr{D}\mathscr{G}_2) \mathbf{u}_Q. \end{aligned} \tag{4.28}$$

Let

$$\tilde{\mathscr{G}_2} = \mathscr{D}\mathscr{G}_2 = \frac{1}{27} \begin{bmatrix} 8 & 2 & -1 \\ 2 & 32 & 2 \\ -1 & 2 & 8 \end{bmatrix}.$$

Then (4.24) follows by the symmetry of $\tilde{\mathscr{G}_2}$. On the other hand, according to Lemma 4.1, in order to prove (4.25), we only need to prove that $\int_{\hat{Q}} \hat{u}_h \widehat{I_h^* \tilde{v}_h}\, \mathrm{d}\hat{\mathbf{x}}$ is a quadratic form on $\mathbf{u}_Q$, which is straightforward by the positive definiteness of matrix $\tilde{\mathscr{G}_2}$.

For any $u_h \in S_h$, let $\alpha = (\mathbf{u}_{x,Q}, \mathbf{u}_{y,Q})$ be a vector with $\mathbf{u}_{x,Q}, \mathbf{u}_{y,Q} \in \mathbb{R}^6$ defined as

$$\begin{aligned} \mathbf{u}_{x,Q} &= (\delta_x u_h(\mathbf{x}_{i+1/2,j}), \delta_x u_h(\mathbf{x}_{i+1/2,j+1/2}), \delta_x u_h(\mathbf{x}_{i+1/2,j+1}), \ldots, \delta_x u_h(\mathbf{x}_{i+1,j+1})), \\ \mathbf{u}_{y,Q} &= (\delta_y u_h(\mathbf{x}_{i,j+1/2}), \delta_y u_h(\mathbf{x}_{i+1/2,j+1/2}), \delta_y u_h(\mathbf{x}_{i+1,j+1/2}), \ldots, \delta_y u_h(\mathbf{x}_{i+1,j+1})). \end{aligned}$$

Since $\tilde{\mathbf{u}}_Q^{\mathrm{T}} = \mathbf{u}_Q^{\mathrm{T}} \mathscr{D} \otimes \mathscr{D}$, a direct calculation yields

$$\tilde{\mathbf{u}}_{x,Q}^{\mathrm{T}} = \mathbf{u}_{x,Q}^{\mathrm{T}} \mathscr{P} \otimes \mathscr{D}, \quad \tilde{\mathbf{u}}_{y,Q}^{\mathrm{T}} = \mathbf{u}_{y,Q}^{\mathrm{T}} \mathscr{P} \otimes \mathscr{D}, \tag{4.29}$$

where

$$\mathscr{P} = \frac{1}{6} \begin{bmatrix} 3+2\sqrt{3} & 3-2\sqrt{3} \\ 3-2\sqrt{3} & 3+2\sqrt{3} \end{bmatrix}.$$

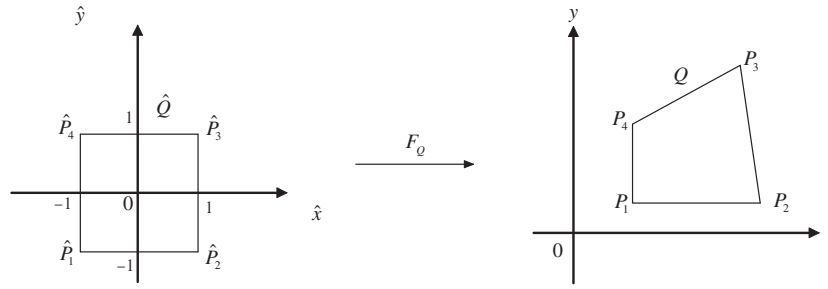

FIG. 3. An invertible bilinear mapping $F_Q : \hat{Q} \to Q$.

Noting that $\mathscr{P}$ and $\mathscr{D}$ are nonsingular, we have

$$|\tilde{u}_h|^2_{1,h,Q} = (\tilde{\mathbf{u}}^{\mathrm{T}}_{x,Q}\tilde{\mathbf{u}}_{x,Q} + \tilde{\mathbf{u}}^{\mathrm{T}}_{y,Q}\tilde{\mathbf{u}}_{y,Q}) \sim (\mathbf{u}^{\mathrm{T}}_{x,Q}\mathbf{u}_{x,Q} + \mathbf{u}^{\mathrm{T}}_{y,Q}\mathbf{u}_{y,Q}) = |u_h|^2_{1,h,Q}.$$

The third step is to prove (4.27), which uses Lemma 4.6 about partitioned matrices that was proved in Yang (2006).

Let P_1, P_2 be two points. We use $|P_1P_2|$ to denote its length. Without loss of generality, we choose $\theta_Q = \angle P_4P_1P_2$ and suppose that Q is a parallelogram with two edges P_1P_2 and P_1P_4 (see Fig. 3). According to Lemma 3.6 in Yang & Liu (2011), under Mesh Assumptions A and B, the difference between the bilinear form $a_h(u_h, I_h^* v_h)$ on a quadrilateral and the one on a parallelogram is $\mathscr{O}(h)|u_h|_1|v_h|_1$. In order to prove (4.27), we only need to prove that $a_{Q,h}(u_h, I_h^*\tilde{u}_h) \gtrsim |u_h|^2_{1,Q}$ holds on the parallelogram.

Let $\kappa = |P_1P_4|/|P_1P_2|$. By (4.9), we can derive

$$a_{Q,h}(u_h, I_h^*\tilde{u}_h) = \tilde{\alpha}\mathscr{A}\alpha^{\mathrm{T}} = \alpha\mathscr{T}\mathscr{A}\alpha^{\mathrm{T}}, \tag{4.30}$$

where

$$\mathscr{T} = \begin{bmatrix} \mathscr{P}\otimes\mathscr{D} & \\ & \mathscr{P}\otimes\mathscr{D} \end{bmatrix}, \quad \mathscr{A} = \frac{|P_1P_2|^2}{m(Q)}\begin{bmatrix} \mathscr{A}_1 & \kappa\mathscr{A}_2 \\ \kappa\mathscr{A}_2 & \kappa^2\mathscr{A}_1 \end{bmatrix}$$

with

$$\mathscr{A}_1 = \frac{1}{6}\begin{bmatrix} 3+2\sqrt{3} & 3-2\sqrt{3} \\ 3-2\sqrt{3} & 3+2\sqrt{3} \end{bmatrix} \otimes \frac{1}{54}\begin{bmatrix} 18-\sqrt{3} & 36-16\sqrt{3} & -\sqrt{3} \\ 2\sqrt{3} & 32\sqrt{3} & 2\sqrt{3} \\ -\sqrt{3} & 36-16\sqrt{3} & 18-\sqrt{3} \end{bmatrix},$$

$$\mathscr{A}_2 = \frac{\cos\theta_Q}{18}\begin{bmatrix} \mathscr{A}_{21} & \mathscr{A}_{22} \\ \mathscr{A}_{23} & \mathscr{A}_{24} \end{bmatrix},$$

$$\mathscr{A}_{21} = \begin{bmatrix} -2\sqrt{3}-1 & 2\sqrt{3}-10 & 3\sqrt{3}-4 \\ -\sqrt{3}-3 & -4\sqrt{3} & -\sqrt{3}+3 \\ 1 & 2\sqrt{3}-2 & \sqrt{3}-2 \end{bmatrix}, \quad \mathscr{R} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix},$$

$$\mathscr{A}_{22} = \mathscr{R}\mathscr{A}_{21}, \quad \mathscr{A}_{23} = \mathscr{A}_{21}\mathscr{R}, \quad \mathscr{A}_{24} = \mathscr{R}\mathscr{A}_{21}\mathscr{R}.$$

Let

$$\tilde{\mathscr{A}} = \mathscr{T}\mathscr{A} = \frac{|P_1P_2|^2}{m(Q)}\begin{bmatrix} \mathscr{P}\otimes\mathscr{D}\mathscr{A}_1 & \kappa\mathscr{P}\otimes\mathscr{D}\mathscr{A}_2 \\ \kappa\mathscr{P}\otimes\mathscr{D}\mathscr{A}_2 & \kappa^2\mathscr{P}\otimes\mathscr{D}\mathscr{A}_1 \end{bmatrix} = \frac{|P_1P_2|^2}{m(Q)}\begin{bmatrix} \tilde{\mathscr{A}}_1 & \kappa\tilde{\mathscr{A}}_2 \\ \kappa\tilde{\mathscr{A}}_2 & \kappa^2\tilde{\mathscr{A}}_1 \end{bmatrix},$$

where $m(Q)$ is the measure of Q. We calculate all the sequential principal minors of the matrices

$$\left(\frac{\tilde{\mathscr{A}}_1+\tilde{\mathscr{A}}_1^{\mathrm{T}}}{2} \pm \frac{\tilde{\mathscr{A}}_2+\tilde{\mathscr{A}}_2^{\mathrm{T}}}{2}\right)$$

and find that the principal minors of the above matrices are positive when $|\cos(\theta_Q)| < 0.97$. Hence the matrices $((\tilde{\mathscr{A}}_1+\tilde{\mathscr{A}}_1^{\mathrm{T}})/2 \pm (\tilde{\mathscr{A}}_2+\tilde{\mathscr{A}}_2^{\mathrm{T}})/2)$ are positive definite. By Lemma 4.6, we know that the matrix $(\tilde{\mathscr{A}}+\tilde{\mathscr{A}}^{\mathrm{T}})/2$ is also positive definite. Its smallest eigenvalue λ_γ depends only on the constant γ. From Mesh Assumption A, we have

$$\lambda_{\min}\left(\frac{\tilde{\mathscr{A}}+\tilde{\mathscr{A}}^{\mathrm{T}}}{2}\right) \geqslant \lambda_\gamma \frac{|P_1P_2|^2}{m(Q)} \geqslant \frac{2\lambda_\gamma}{\sigma}. \tag{4.31}$$

Accordingly the following holds for a parallelogram

$$a_{Q,h}(u_h, I_h^*\tilde{u}_h) = \alpha\frac{\mathscr{T}\mathscr{A}+(\mathscr{T}\mathscr{A})^{\mathrm{T}}}{2}\alpha^{\mathrm{T}} = \alpha\frac{\tilde{\mathscr{A}}+\tilde{\mathscr{A}}^{\mathrm{T}}}{2}\alpha^{\mathrm{T}} \geqslant \frac{2\lambda_\gamma}{\sigma}|u_h|_{1,h,Q}^2.$$

Therefore, $a_h(u_h, I_h^*\tilde{u}_h) \gtrsim |u_h|_1^2$ holds for quadrilateral meshes that satisfy Mesh Assumptions A and B. □

LEMMA 4.6 The matrix $\left[\begin{smallmatrix} \mathscr{A} & \kappa\mathscr{A} \\ \kappa\mathscr{A}^{\mathrm{T}} & \kappa^2\mathscr{A} \end{smallmatrix}\right]$ is positive definite if and only if matrices $\mathscr{A} \pm \frac{1}{2}(\mathscr{A}+\mathscr{A}^{\mathrm{T}})$ are positive definite.

LEMMA 4.7 Assume Mesh Assumptions A and B are satisfied. For any $u_h, v_h \in S_h$, we have

$$|(u_h, I_h^*\tilde{v}_h) - (v_h, I_h^*\tilde{u}_h)| \lesssim h\|u_h\|_0\|v_h\|_0. \tag{4.32}$$

Proof. For any $Q \in \Omega_h$, a change of variables in multiple integrals gives us

$$\int_Q u_h I_h^*\tilde{v}_h \mathrm{d}\mathbf{x} = \int_{\hat{Q}} J_{F_Q}\hat{u}_h\widehat{I_h^*\tilde{v}_h}\mathrm{d}\hat{\mathbf{x}}.$$

By (4.24), we have

$$J_{F_Q}(0,0)\int_{\hat{Q}}\hat{u}_h\widehat{I_h^*\tilde{v}_h}\,\mathrm{d}\hat{\mathbf{x}} - J_{F_Q}(0,0)\int_{\hat{Q}}\hat{v}_h\widehat{I_h^*\tilde{u}_h}\,\mathrm{d}\hat{\mathbf{x}} = 0. \tag{4.33}$$

It was proved in Li & Li (1999) under Mesh Assumptions A and B that

$$|J_{F_Q} - J_{F_Q}(0,0)| \lesssim h_Q^3.$$

Combining the above estimates and using (4.28), we obtain

$$\begin{aligned}\left|\int_Q u_h I_h^* \tilde{v}_h \,\mathrm{d}\mathbf{x} - \int_Q v_h I_h^* \tilde{u}_h \,\mathrm{d}\mathbf{x}\right| &\leqslant \left|\int_Q u_h I_h^* \tilde{v}_h \,\mathrm{d}\mathbf{x} - J_{F_Q}(0,0)\int_{\hat{Q}} \hat{u}_h \widehat{I_h^* \tilde{v}_h} \,\mathrm{d}\hat{\mathbf{x}}\right| \\ &\quad + \left|\int_Q v_h I_h^* \tilde{u}_h \,\mathrm{d}\mathbf{x} - J_{F_Q}(0,0)\int_{\hat{Q}} \hat{v}_h \widehat{I_h^* \tilde{u}_h} \,\mathrm{d}\hat{\mathbf{x}}\right| \\ &\lesssim h_Q^3 \|\hat{u}_h\|_{0,\hat{Q}} \|\hat{v}_h\|_{0,\hat{Q}} \lesssim h_Q \|u_h\|_{0,Q} \|v_h\|_{0,Q}.\end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned}|(u_h, I_h^* \tilde{v}_h) - (v_h, I_h^* \tilde{u}_h)| &\leqslant \sum_{Q\in\Omega_h} \left|\int_Q u_h I_h^* \tilde{v}_h \,\mathrm{d}\mathbf{x} - \int_Q v_h I_h^* \tilde{u}_h \,\mathrm{d}\mathbf{x}\right| \\ &\lesssim \sum_{Q\in\Omega_h} h_Q \|u_h\|_{0,Q} \|v_h\|_{0,Q} \lesssim h \|u_h\|_0 \|v_h\|_0.\end{aligned}$$

□

THEOREM 4.2 Let u be the solution of (3.8) and u_h the numerical solution of the semidiscrete finite-volume scheme (3.9). Assume that $u \in L^\infty(0,T;H^4) \cap H^1(0,T;H^4)$ and Mesh Assumptions A, B and C are satisfied. Then

$$\|u(t) - u_h(t)\|_0 + h|u(t) - u_h(t)|_1 + \left(\int_0^t |I_h u - u_h|_1^2 \,\mathrm{d}t\right)^{1/2} \lesssim h^3, \quad 0 \leqslant t \leqslant T. \tag{4.34}$$

Proof. We decompose the error as $u_h - u = \xi - \eta$, where $\xi = u_h - R_h u$ and $\eta = u - R_h u$. We have the following error equation:

$$(\xi_t, I_h^* v_h) + a_h(\xi, I_h^* v_h) = (\eta_t, I_h^* v_h). \tag{4.35}$$

Taking $v_h = \xi_t$ in (4.35) leads to

$$\begin{aligned}\|\xi_t\|_0^2 &= (\eta_t, I_h^* \xi_t) - a_h(\xi, I_h^* \xi_t) \\ &\leqslant \|\eta_t\|_0 \|I_h^* \xi_t\|_0 + C|\xi|_1 |\xi_t|_1 \\ &\leqslant \|\eta_t\|_0 \|I_h^* \xi_t\|_0 + Ch^{-1}|\xi|_1 \|\xi_t\|_0,\end{aligned}$$

where an inverse estimate is used for the last inequality. Utilizing the norm equivalence, we have

$$\|\xi_t\|_0 \lesssim \|\eta_t\|_0 + h^{-1}|\xi|_1. \tag{4.36}$$

On the other hand, we take $v_h = \tilde{\xi}$ in (4.35) to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\xi, I_h^* \tilde{\xi}) + a_h(\xi, I_h^* \tilde{\xi}) = \frac{1}{2}((\xi, I_h^* \tilde{\xi}_t) - (\xi_t, I_h^* \tilde{\xi})) + (\eta_t, I_h^* \tilde{\xi}).$$

By Lemmas 4.5 and 4.7, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\xi, I_h^*\tilde{\xi}) + |\xi|_1^2 &\lesssim h\|\xi\|_0\|\xi_t\|_0 + \|\eta_t\|_0\|I_h^*\tilde{\xi}\|_0 \\ &\lesssim h\|\xi\|_0\|\xi_t\|_0 + \|\eta_t\|_0\|\xi\|_0. \end{aligned} \tag{4.37}$$

Combining (4.36) and (4.37) together yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\xi, I_h^*\tilde{\xi}) + |\xi|_1^2 \lesssim h\|\xi\|_0\|(\|\eta_t\|_0 + h^{-1}|\xi|_1) + \|\eta_t\|_0\|\xi\|_0. \tag{4.38}$$

Integrating (4.38) on $[0, t]$, noting that $\xi(0) = 0$, and using Lemma 4.5, we have

$$\begin{aligned}\|\xi(t)\|_0^2 + \int_0^t |\xi|_1^2\,\mathrm{d}t &\lesssim \int_0^t ((h+1)\|\xi\|_0\|\eta_t\|_0 + \|\xi\|_0|\xi|_1)\,\mathrm{d}t \\ &\leqslant C\int_0^t \|\xi\|_0^2\,\mathrm{d}t + C\int_0^t \|\eta_t\|_0^2\,\mathrm{d}t + \epsilon\int_0^t |\xi|_1^2\,\mathrm{d}t.\end{aligned}$$

Then taking $\epsilon < 1$ and using the Gronwall's inequality yield

$$\|\xi(t)\|_0^2 + \int_0^t |\xi|_1^2\,\mathrm{d}t \lesssim \int_0^t \|\eta_t\|_0^2\,\mathrm{d}t. \tag{4.39}$$

By the inverse estimate,

$$h|\xi|_1 \lesssim \|\xi\|_0 \lesssim \left(\int_0^t \|\eta_t\|_0^2\,\mathrm{d}t\right)^{1/2}. \tag{4.40}$$

From (4.39), (4.40) and Theorem 4.1, we have

$$\begin{aligned}&\|u - u_h\|_0 + h|u - u_h|_1 + \left(\int_0^t |I_h u - u_h|_1^2\,\mathrm{d}t\right)^{1/2} \\ &\quad\leqslant \|\xi\|_0 + \|u - R_h u\|_0 + h(|\xi|_1 + |u - R_h u|_1) + \left(2\int_0^t |\xi|_1^2 + |I_h u - R_h u|_1^2\,\mathrm{d}t\right)^{1/2} \\ &\quad\lesssim h^3\left(\|u\|_{L^\infty(H^4)} + \left(\int_0^t \|u_t\|_4^2\,\mathrm{d}t\right)^{1/2}\right),\end{aligned}$$

which gives the desired result. □

THEOREM 4.3 Let u be the solution of (3.8) and u_h^n the numerical solution of the fully discrete finite-volume scheme (3.12). Assume that $u \in L^\infty(0, T; H^4) \cap H^1(0, T; H^4) \cap H^3(0, T; L^2)$ and Mesh Assumptions A, B and C are satisfied. Then we have the following error estimate:

$$\|u^M - u_h^M\|_0 + h|u^M - u_h^M|_1 + \left(\Delta t\sum_{n=1}^{M} |I_h u^{n,1/2} - u_h^{n,1/2}|_1^2\right)^{1/2} \lesssim h^3 + \Delta t^2, \quad 0 \leqslant M \leqslant N. \tag{4.41}$$

Proof. Let $\xi = u_h - R_h u$ and $\eta = u - R_h u$. It is obvious that $\xi^0 = 0$. We have the following error equation:

$$(\bar{\partial}\xi^n, I_h^* v_h) + a_h(\xi^{n,1/2}, I_h^* v_h) = (\omega^n, I_h^* v_h), \quad n \geqslant 1, \tag{4.42}$$

where

$$\omega^n = \bar{\partial}\eta^n + u_t^{n,1/2} - \bar{\partial}u^n.$$

We choose $v_h = \bar{\partial}\xi^n$ in (4.42) and take a similar argument as that in Theorem 4.2 to obtain

$$\|\bar{\partial}\xi^n\|_0 \lesssim \|\omega^n\|_0 + h^{-1}|\xi^{n,1/2}|_1. \tag{4.43}$$

On the other hand, we choose $v_h = 2\Delta t\tilde{\xi}^{n,1/2}$ in (4.42) to obtain

$$\begin{aligned}(\xi^n, I_h^*\tilde{\xi}^n) - (\xi^{n-1}, I_h^*\tilde{\xi}^{n-1}) + 2\Delta t a_h(\xi^{n,1/2}, I_h^*\tilde{\xi}^{n,1/2})\\ = (\xi^{n-1}, I_h^*\tilde{\xi}^n) - (\xi^n, I_h^*\tilde{\xi}^{n-1}) + 2\Delta t(\omega^n, I_h^*\tilde{\xi}^{n,1/2}),\end{aligned}$$

where we use Lemma 4.7 and (4.43) to find

$$\begin{aligned}(\xi^{n-1}, I_h^*\tilde{\xi}^n) - (\xi^n, I_h^*\tilde{\xi}^{n-1}) &= (\xi^{n-1}, I_h^*(\tilde{\xi}^n - \tilde{\xi}^{n-1}) - (\xi^n - \xi^{n-1}, I_h^*\tilde{\xi}^{n-1})\\ &\leqslant C\Delta t h\|\bar{\partial}\xi^n\|_0\|\xi^{n-1}\|_0 \leqslant C\Delta t(h\|\omega^n\|_0 + |\xi^{n,1/2}|_1)\|\xi^{n-1}\|_0.\end{aligned}$$

Therefore, summing from $n = 1$ to M yields

$$\begin{aligned}&(\xi^M, I_h^*\tilde{\xi}^M) + 2\Delta t\sum_{n=1}^{M} a_h(\xi^{n,1/2}, I_h^*\tilde{\xi}^{n,1/2})\\ &\leqslant C\Delta t\sum_{n=1}^{M}(h\|\omega^n\|_0 + |\xi^{n,1/2}|_1)\|\xi^{n-1}\|_0 + 2\Delta t\sum_{n=1}^{M}(\omega^n, I_h^*\tilde{\xi}^{n,1/2})\\ &\leqslant C\Delta t\sum_{n=1}^{M}\|\omega^n\|_0^2 + C\Delta t\sum_{n=1}^{M}\|\xi^{n-1}\|_0^2 + \epsilon\Delta t\sum_{n=1}^{M}|\xi^{n,1/2}|_1^2.\end{aligned}$$

Using Lemma 4.5, taking ϵ small enough and applying the discrete Gronwall's inequality, we have

$$\|\xi^M\|_0^2 + \Delta t\sum_{n=1}^{M}|\xi^{n,1/2}|_1^2 \lesssim \Delta t\sum_{n=1}^{M}\|\omega^n\|_0^2.$$

The Taylor's expansion with the remainder term in the integral form gives

$$\begin{aligned}\Delta t\sum_{n=1}^{M}\|\omega^n\|_0^2 &\leqslant 2\Delta t\sum_{n=1}^{M}(\|\bar{\partial}\eta^n\|_0^2 + \|u_t^{n,1/2} - \bar{\partial}u^n\|_0^2)\\ &\leqslant C\left(h^6\int_0^{t_M}\|u_t\|_4^2\,\mathrm{d}t + \Delta t^4\int_0^{t_M}\|u_{ttt}\|_0^2\,\mathrm{d}t\right).\end{aligned}$$

Hence,

$$\|\xi^M\|_0^2 + \Delta t \sum_{n=1}^{M} |\xi^{n,1/2}|_1^2 \lesssim h^6 \int_0^{t_M} \|u_t\|_4^2 \,\mathrm{d}t + \Delta t^4 \int_0^{t_M} \|u_{ttt}\|_0^2 \,\mathrm{d}t.$$

Finally, the desired result follows by an inverse estimate and a triangle inequality. □

REMARK 4.2 Note from Theorems 4.1–4.3 that our error estimates are optimal with respect to the degree of approximating polynomials used, but suboptimal with respect to the solution regularity. There are two main reasons. First, to obtain optimal order errors, finite-volume element methods almost need higher regularity assumptions on the exact solution than the finite element methods (even for linear cases), see, e.g., Chou & Ye (2007), Xu & Zou (2009) and Yang (2006). Secondly. the analysis is based on the superapproximation result for $a_h(u - I_h u, I_h^* v_h)$, which convergences one order higher than the optimal rate. The errors in the H^1- and L^2-norms can be regarded as a by-product of this superconvergence result. This technique successfully avoids the difficulty of the duality argument for higher order finite-volume methods but needs higher regularity on the exact solution.

5. Numerical experiments

In this section, we present numerical results to illustrate the theoretical findings in this paper.

We test the finite-volume methods for elliptic and parabolic problems on a family of quadrilateral meshes that are obtained by perturbing a set of rectangular meshes randomly. The rectangular meshes have $M = 8, 16, 32, 64, 128$ partitions, respectively, in both x, y-directions. The quadrilateral meshes are then generated by

$$x_{i,j} = \frac{\pi}{M} i + \frac{\pi}{4M} \sin(j)\,\mathrm{rand}(), \quad y_{i,j} = \frac{\pi}{M} j + \frac{\pi}{4M} \sin(i)\,\mathrm{rand}(), \quad 1 \leqslant i, j \leqslant M - 1,$$

where $\sin(i), \sin(j)$ use the radian unit for angles and `rand()` generates a random number in $(0, 1)$.

The preconditioned bi-conjugate gradient stabilized method (BiCGStab) is employed to solve the nonsymmetric discrete linear systems. The tolerance for residuals is set as 10^{-12} and the simple diagonal preconditioning is used. The numerical order of convergence is then measured by comparing the computed errors on two successive mesh levels.

EXAMPLE 1 (An Elliptic Problem) The domain is $\Omega = [0, \pi]^2$, the permeability coefficient $a(x, y) = (x + 1)^2 + y^2$, the exact solution $u(x, y) = \sin(x)\sin(y)$. This problem was tested in Yang (2006).

Table 1 shows a third-order convergence in L^2-norm of the error $u - u_h$ and also a third-order convergence in the H^1-seminorm of $h(u - u_h)$. One can also observe a third-order superapproximation in $|I_h u - u_h|_{1,h}$, which is equivalent to $|I_h u - u_h|_1$ by Lemma 4.1.

For comparison, Table 2 tabulates numerical results for the elliptic problem in Example 1 by using the finite-volume scheme based on the Simpson quadrature points. One can observe only a second-order convergence in $|I_h u - u_h|_{1,h}$.

EXAMPLE 2 (A Parabolic Problem) The domain is $\Omega = [0, \pi]^2$, the permeability coefficient $a(x, y) = (x + 1)^2 + y^2$, the exact solution $u(x, y, t) = e^{-0.1t}\sin(x)\sin(y)$ and the final time $T = 0.2$. The right-hand side $f(x, y, t)$ is computed accordingly. This example was tested in Yang & Liu (2011).

In Table 3, we present numerical results obtained from using the Crank–Nicholson scheme for temporal discretization. We take a fixed but relatively small time step $\Delta t = T/256$ so that the spatial errors dominate. One can observe third-order convergence in h for $\max_{0 \leqslant n \leqslant N} \|u^n - u_h^n\|_0$ and

TABLE 1 *Example 1: Numerical errors and convergence rates for the finite-volume scheme using Barlow points for the elliptic problem on quadrilateral meshes with $h = \pi/M$*

M	$\|u - u_h\|_0$	Rate	$\|u - u_h\|_1$	Rate	$\|I_h u - u_h\|_{1,h}$	Rate
8	4.942E−4	—	7.704E−3	—	1.275E−3	—
16	5.231E−5	3.23	1.691E−3	2.18	7.223E−5	4.14
32	6.173E−6	3.08	4.051E−4	2.06	5.691E−6	3.66
64	7.595E−7	3.02	1.001E−4	2.01	5.948E−7	3.25
128	9.456E−8	3.00	2.495E−5	2.00	7.073E−8	3.07

TABLE 2 *Example 1: Numerical errors and convergence rates for the finite-volume scheme using Simpson points for the elliptic problem on quadrilateral meshes with $h = \pi/M$*

M	$\|u - u_h\|_0$	Rate	$\|u - u_h\|_1$	Rate	$\|I_h u - u_h\|_{1,h}$	Rate
8	2.119E−3	—	8.272E−3	—	7.785E−3	—
16	4.537E−4	2.22	1.810E−3	2.19	1.618E−4	2.26
32	1.077E−4	2.07	4.329E−4	2.06	3.765E−4	2.10
64	2.650E−5	2.02	1.069E−4	2.01	9.203E−5	2.03
128	6.592E−6	2.00	2.664E−5	2.00	2.285E−5	2.00

TABLE 3 *Example 2: Numerical errors and convergence rates for the parabolic problem on quadrilateral meshes using the Crank–Nicholson scheme and Barlow points with fixed $\Delta t = T/256$ and varying $h = \pi/M$*

M	$\max_{0\leqslant n\leqslant N} \|u^n - u_h^n\|_0$	Rate	$\max_{0\leqslant n\leqslant N} \|u^n - u_h^n\|_1$	Rate	$\|I_h u - u_h\|_E$	Rate
8	4.999E−4	—	7.714E−3	—	5.525E−4	—
16	5.239E−5	3.25	1.691E−3	2.18	3.165E−5	4.12
32	6.175E−6	3.08	4.051E−4	2.06	2.531E−6	3.64
64	7.595E−7	3.02	1.001E−4	2.01	2.718E−7	3.21
128	1.010E−7	2.91	2.495E−4	2.00	4.853E−8	2.48

$\max_{0\leqslant n\leqslant N} h|u^n - u_h^n|_1$, as proved in Theorem 4.3. For $M = 128$, the L^2 and energy error convergence rates drop a bit. This is because the dominance of the spatial errors becomes weaker as M increases.

In Table 4, we present numerical results obtained from using the third-order backward differentiation formula (*BDF3*) (Iserles, 1996) for temporal discretization. It is reflected in Table 4 that

$$\|u^M - u_h^M\|_0 + h|u^M - u_h^M|_1 \lesssim h^3 + \Delta t^3, \quad 0 \leqslant M \leqslant N,$$

when $\Delta t = \mathscr{O}(h)$.

In Tables 3 and 4, we consider also a discrete temporal energy norm

$$|I_h u - u_h|_E := \left(\Delta t \sum_{n=1}^{N} |I_h u^n - u_h^n|_{1,h}^2 \right)^{1/2}.$$

TABLE 4 *Example 2: Numerical errors and convergence rates for the parabolic problem on quadrilateral meshes using the BDF3 scheme and Barlow points with* $h = \pi/M$, $\Delta t = (0.1/\pi)h, N = 2M$

M	$\max_{0\leqslant n\leqslant N} \|u^n - u_h^n\|_0$	Rate	$\max_{0\leqslant n\leqslant N} \|u^n - u_h^n\|_1$	Rate	$\|I_h u - u_h\|_E$	Rate
8	4.957E−4	—	7.694E−3	—	5.521E−4	—
16	5.232E−5	3.24	1.689E−3	2.18	3.156E−5	4.12
32	6.171E−6	3.08	4.050E−4	2.06	2.506E−6	3.65
64	7.594E−7	3.02	1.001E−4	2.01	2.640E−7	3.24
128	9.458E−8	3.00	2.495E−5	2.00	3.256E−8	3.01

Table 3 uses Crank–Nicholson for temporal discretization, whereas Table 4 uses BDF3. One can clearly observe better convergence rates in the energy norm from using the *Barlow* points than those from using *Simpson* points (see Yang & Liu, 2011, Table 2).

6. Concluding remarks

The usefulness of quadrilateral meshes for developing numerical methods for partial differential equations has been demonstrated in Arnold *et al.* (2002), Flemisch & Wohlmuth (2007), Li & Li (1999) and Schroll & Svensson (2006). Shape parameters for characterizing quality of quadrilateral meshes were discussed in Chou & He (2002) and Yang & Liu (2011). Mesh Assumption A and B are used for obtaining optimal error estimates, whereas Mesh Assumption C is used for deriving the superconvergence.

The new finite-volume schemes developed in this paper have optimal convergence rates, thanks to the Barlow points. Barlow points were originally discovered as optimal stress sampling points in finite bar elements (Barlow, 1976). It was then realized that Barlow points are the same as Gaussian points for linear and quadratic bar elements, but different for cubic bar elements (Júnior, 2008). The discrepancy has motivated alternative approaches, e.g., the variational approach, the best-fit method that leads to discovery of Prathap points and establishment of the relationships among Barlow points, Gaussian points and Prathap points for one-dimensional finite elements (Rajendran, 2009). Barlow points can be established for rectangular elements through tensor products and for quadrilateral elements via bilinear transformations. However, the study in Rajendran & Liew (2003) implies that the effectiveness of Barlow points, Gaussian points, and Prathap points for linear and quadratic finite (volume) elements on triangular meshes seems limited. On the other hand, construction of higher order Lagrange and Hermite-type finite-volume element methods on triangular elements are investigated in Chen *et al.* (2011) in a general framework. Development of higher order finite-volume methods remains an active and promising research front.

Funding

The first author is supported by Shandong Province Higher Educational Science and Technology Program (J10LA01), Shandong Province Natural Science Foundation (ZR2010AQ020) and National Natural Science Foundation of China (11201405). The third author is partially supported by GRF grant of Hong Kong (No. PolyU 501709), JRI of PolyU and NSERC (Canada).

REFERENCES

ARNOLD, D., BOFFI, D. & FALK, R. (2002) Approximation by quadrilateral finite elements. *Math. Comput.*, **71**, 909–922.

Barlow, J. (1976) Optimal stress locations in finite element models. *Int. J. Numer. Methods Eng.*, **10**, 243–251.

Brenner, S. C. & Scott, L. R. (2008) *The Mathematical Theory of Finite Element Methods*, 3rd edn. New York: Springer.

Cai, Z., Douglas Jr, J., & Park, M. (2003) Development and analysis of higher order finite volume methods over rectangles for elliptic equations. *Adv. Comput. Math.*, **19**, 3–33.

Chen, L. (2010) A new class of high order finite volume methods for second order elliptic equations. *SIAM J. Numer. Anal.*, **47**, 4021–4043.

Chen, Z., Wu, J. & Xu, Y. (2011) Higher-order finite volume methods for elliptic boundary value problems. *Adv. Comput. Math.*, doi:10.1007/s10444-011-9201-8.

Chou, S. H. & He, S. (2002) On the regularity and uniformness conditions on quadrilateral grids. *Comput. Meth. Appl. Mech. Engrg.*, **191**, 5149–5158.

Chou, S. H. & Ye, X. (2007) Unified analysis of finite volume methods for second order elliptic problems. *SIAM J. Numer. Anal.*, **45**, 1639–1653.

Ewing, R. E., Liu, M. & Wang, J. (1999) Superconvergence of mixed finite element approximations over quadrilaterals. *SIAM J. Numer. Anal.*, **36**, 772–787.

Flemisch, B. & Wohlmuth, B. (2007) Stable Lagrange multipliers for quadrilateral meshes of curved interfaces in 3D. *Comput. Meth. Appl. Mech. Engrg.*, **196**, 1589–1602.

Gao, G. & Wang, T. (2010) Cubic superconvergent finite volume element method for one-dimensional elliptic and parabolic equations. *J. Comput. Appl. Math.*, **2010**, 2285–2301.

Hackbusch, W. (1989) On first and second order box schemes. *Computing*, **41**, 277–296.

Hyman, J. M., Knapp, R. & Scovel, J. C. (1992) High order finite volume approximations of differential operators on nonuniform grids. *Physica D*, **60**, 112–138.

Iserles, A. (1996) *A First Course in the Numerical Analysis of Differential Equations*. Cambridge: Cambridge University Press.

Júnior, D. S. P. (2008) Studies on Barlow points, Gauss points and superconvergent points in 1D with Lagrangian and Hermitian finite element basis. *Comput. Appl. Math.*, **27**, 275–303.

Li, R., Chen, Z. & Wu, W. (2000) *Generalized Difference Methods for Differential Equations: Numerical Analysis of Finite Volume Methods*. New York: Marcel Dekker.

Li, Y. & Li, R. (1999) Generalized difference methods on arbitrary quadrilateral networks. *J. Comput. Math.*, **17**, 653–672.

Liebau, F. (1996) The finite volume element method with quadratic basis functions. *Computing*, **57**, 281–299.

Plexousakis, M. & Zouraris, G. E. (2004) On the construction and analysis of high order locally conservative finite volume-type methods for one-dimensional elliptic problems. *SIAM J. Numer. Anal.*, **42**, 1226–1260.

Rajendran, S. (2009) Optimal stress recovery points for higher-order bar elements by the Prathap's best-fit method. *Comm. Numer. Meth. Engrg.*, **25**, 864–881.

Rajendran, S. & Liew, K. M. (2003) Optimal stress sampling points of plane triangular elements for patch recovery of nodal stresses. *Int. J. Numer. Meth. Engrg.*, **58**, 579–607.

Schroll, H. J. & Svensson, F. (2006) A bi-hyperbolic finite volume method on quadrilateral meshes. *J. Sci. Comput.*, **26**, 237–260.

Thomée, V. (2006) *Galerkin Finite Element Methods for Parabolic Problems*. Berlin: Springer.

Wang, T. & Gu, Y. (2010) Superconvergent biquadratic finite volume element method for two-dimensional Poisson's equations. *J. Comput. Appl. Math.*, **234**, 447–460.

Xu, J. C. & Zou, Q. (2009) Analysis of linear and quadratic simplicial finite volume methods for elliptic equations. *Numer. Math.*, **111**, 469–492.

Yang, M. (2006) A second-order finite volume element method on quadrilateral meshes for elliptic equations. *ESAIM Math. Model. Numer. Anal.*, **40**, 1053–1068.

Yang, M. & Liu, J. (2011) A quadratic finite volume element method for parabolic problems on quadrilateral meshes. *IMA J. Numer. Anal.*, **31**, 1038–1061.

Yang, M., Liu, J. & Chen, C. (2009) Error estimation of a quadratic finite volume method on right quadrangular prism grids. *J. Comput. Appl. Math.*, **229**, 274–282.

Yu, C. & Li, Y. (2010) Biquadratic element finite volume method based on optimal stress points for solving Poisson equations. *Math. Numer. Sin.*, **32**, 59–74 (in Chinese).

Yu, C. & Li, Y. (2011) Biquadratic finite volume element methods based on optimal stress points for parabolic problems. *J. Comput. Appl. Math.*, **236**, 1065–1068.

Zlámal, M. (1978) Superconvergence and reduced integration in the finite element method. *Math. Comp.*, **143**, 663–685.