A quadratic finite volume element method for parabolic problems on quadrilateral meshes

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In this paper we utilize affine biquadratic elements and a two-step temporal discretization to develop a finite volume element method for parabolic problems on quadrilateral meshes. The method is proved to have an optimal order convergence rate in $L^2(0, T; H^1(\Omega))$ under the ‘asymptotically parallelogram’ mesh assumption. Numerical experiments that corroborate the theoretical analysis are also presented.

Keywords: error estimation; finite volume element methods; parabolic equations; quadrilateral meshes; second-order accuracy.

1. Introduction

In this paper we develop a second-order finite-volume method for the following model parabolic initial boundary-value problem:

$$
\begin{aligned}
&u_t - \nabla \cdot (a(x) \nabla u) = f(x, t), \quad (x, t) \in \Omega \times (0, T], \\
&u = 0, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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 Hackbusch (1989), Morton (1998), Ewing et al. (2002), Carstensen et al. (2005), Nicaise & Djadel (2005), Ye (2006) and Sinha & Geiser (2007) (also the references therein) for more details. Compared to finite-difference and finite-element methods, finite-volume methods are usually easier to implement and offer flexibility in handling complicated domain geometries. More importantly, the methods ensure local mass conservation, a highly desirable property in many applications.

Piecewise linear finite-volume methods have been extensively studied for various parabolic problems. The main idea is to treat the linear finite-volume methods as ‘small perturbations’ of the linear finite-element methods. Consequently, techniques (such as a posteriori error estimates and superconvergence) for the finite elements can also be used for the finite-volume methods. In Chou & Li (2000) a unified approach was presented to derive error estimates in the $L^2$, $H^1$- and $L^\infty$-norms. Later, error estimates and superconvergence results in the $L^p$-norm ($2 \leq p < \infty$) were obtained by Chou et al. (2003). Finite volume element methods for parabolic problems in convex polygonal domains were studied by Chatzipantelidis et al. (2004) and error estimates in the $H^1$, $L^2$- and $L^\infty$-norms under limited regularities of exact solutions were established. In order to solve the discrete equations more efficiently, Rui (2002) and Ma et al. (2003) constructed several symmetric finite-volume schemes. Ohlberger & Rohde (2002) obtained a posteriori error estimates in the $L^1$-norm for finite-volume approximations of weakly coupled convection-dominated parabolic systems.

Compared to linear finite-volume methods, quadratic and higher-order elements differ considerably from the corresponding finite-element methods. The development of higher-order finite-volume methods is a challenging task and has been attracting many researchers’ attention in recent years. Plexousakis & Zouraris (2004) performed an analysis of higher-order finite-volume methods for one-dimensional elliptic problems. Cai et al. (2003) presented several higher-order finite-volume schemes from mixed finite-element methods over rectangular meshes for elliptic problems. Based on different dual partitions, two types of quadratic simplicial finite-volume schemes were discussed in Liebau (1996) and Li et al. (2000) for two-dimensional elliptic problems, and both schemes were shown to be second-order accurate in the $H^1$-norm. Xu & Zou (2009) combined the above two schemes together and gave an analysis under some weak conditions on grids. Error estimation of quadratic quadrilateral finite-volume methods for elliptic problems was studied in Yang (2006). The analysis was then extended to three-dimensional right quadrangular prism meshes in Yang et al. (2009).

The aforementioned papers on higher-order finite-volume schemes focused on stationary problems. To the best of the authors’ knowledge, little progress has been made on time-dependent problems up to now. The main difficulty is due to the fact that the inner product $(\cdot, I^h_\star \cdot)$ and the weak form $a_h(\cdot, I^h_\star \cdot)$ (see Section 2) of higher-order finite-volume methods are not symmetric even for constant coefficient problems. Therefore it is difficult to handle temporal terms in theoretical analysis. How to control non-symmetry of the related bilinear forms is generally still not clear.

It is well known that quadrilateral meshes have simpler mesh data structures than triangular meshes but are more flexible than rectangular meshes in handling complicated domain geometries. Shown in Fig. 1 are two examples of domains for which quadrilateral meshes are preferred: a compression channel and a triangular obstacle in a flow (see Schroll & Svensson, 2006).

The purpose of this paper is to develop a second-order finite volume element method on quadrilateral meshes for parabolic problems. It is a continuation of our efforts in Yang (2006) and Yang et al. (2009) for two-dimensional and three-dimensional elliptic problems. However, the dual partition used in Yang (2006) for elliptic problems no longer works for parabolic problems (Remark 3.1). Here we introduce a new dual partition for two-dimensional parabolic problems. By exploring properties of affine mappings, we manage to control non-symmetry of the bilinear form $(\cdot, I^h_\star \cdot)$ (Lemma 3.5) and ensure coercivity of the bilinear form $a_h(\cdot, I^h_\star \cdot)$ (Lemma 3.8). Our new method also features a well-balanced combination
of a quadratic spatial approximation and a two-step temporal discretization. As an implicit method, it is unconditionally stable (Theorem 4.1), allows relatively large time steps ($\Delta t = \mathcal{O}(h)$) and can generate numerical solutions that have a second-order accuracy in the energy norm (Theorem 4.2).

The rest of this paper is organized as follows. In Section 2 we introduce some notation and establish a finite volume element scheme for the parabolic initial boundary-value problem (1.1). In Section 3 we prove some auxiliary results about quadrilateral meshes and the bilinear forms. A second-order error estimate in the energy norm is proved in Section 4 under certain regularity conditions. Section 5 presents numerical experiments to illustrate the theoretical analysis. The paper is concluded with some remarks in Section 6.

Throughout this paper we use $C$ (with or without subscripts) to denote a generic positive constant that is independent of discretization parameters.

2. A fully discrete finite-volume scheme on quadrilateral meshes

2.1 Notation

We shall use the standard notation for the Sobolev spaces $W^{m,p}(\Omega)$ with the norm $\| \cdot \|_{m,p,\Omega}$ and the seminorm $| \cdot |_{m,p,\Omega}$. We also denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and omit the index $p = 2$ and the domain $\Omega$ when there is no ambiguity, i.e., $\|u\|_{m,p} = \|u\|_{m,p,\Omega}$ and $\|u\|_m = \|u\|_{m,2,\Omega}$. The same convention is adopted for the seminorms.

Let $\Omega_h = \{ Q \}$ be a quadrilateral partition of $\Omega$, where any two closed quadrilaterals share a common edge, vertex or nothing. Let $\hat{Q} = (0, 1)^2$ be the reference element in the $\hat{x}\hat{y}$-plane. For each element $Q \in \Omega_h$ there exists a bijective bilinear mapping $F_Q : \hat{Q} \rightarrow Q$ satisfying (see Fig. 2)

$$F_Q(\hat{P}_i) = P_i, \quad 1 \leq i \leq 4.$$ 

Let $\mathcal{J}_{F_Q}$ be the Jacobian matrix of $F_Q$ at $\hat{x}$ and $J_{F_Q} = \det \mathcal{J}_{F_Q}$. Accordingly, $\mathcal{J}_{F_Q^{-1}}$ is the Jacobian matrix of $F_Q^{-1}$ at $x$ and $J_{F_Q^{-1}} = \det \mathcal{J}_{F_Q^{-1}}$. Based on the partition $\Omega_h$, we define $S_h$ as the standard conforming finite-element space of piecewise affine biquadratic functions

$$S_h = \{ v \in H^1_0(\Omega) : v|_Q = \hat{v} \circ F_Q^{-1}, \hat{v}|_{\hat{Q}} \text{ is biquadratic } \forall Q \in \Omega_h; v|_{\partial \Omega} = 0 \}. \quad \text{(2.1)}$$

Before developing the finite volume element method, we make some assumptions on quadrilateral meshes. For any $Q \in \Omega_h$ let $h_Q$ be its diameter, let $h'_Q$ be the smallest length of the edges and let $\theta_Q$ be any interior angle. We set $h = \max_{Q \in \Omega_h} h_Q$. 

FIG. 1. Two examples that favour quadrilateral meshes.
Mesh assumption A. The partition $\Omega_h = \{Q\}$ is regular, that is, there exist two positive constants $\sigma$ and $\gamma$ such that

$$h_Q / h'_Q \leq \sigma, \quad |\cos \theta_Q| \leq \gamma < 1 \quad \forall Q \in \Omega_h.$$  

(2.2)

Mesh assumption B. The quadrilateral meshes are ‘asymptotically parallelogram’, that is, for $Q \in \Omega_h$ one has

$$\vartheta_Q = O(h_Q),$$

(2.3)

where $\vartheta_Q = \max(|\pi - \theta_1|, |\pi - \theta_2|)$, $\theta_1$ is the angle between the outward normals of two opposite sides of $Q$ and $\theta_2$ is the angle between the outward normals of the other two sides.

Remark 2.1 The asymptotically parallelogram assumption has been adopted by many authors, although it takes several different forms in the literature. In Arnold et al. (2002) it was called ‘asymptotically parallelogram’ and defined through the maximal angle difference of the outward normals. It is referred to as ‘$h^2$-parallelogram’ and defined through the edge deviation in Ewing et al. (1999). A definition based on the distance of the two diagonal midpoints was adopted by Süli (1992). A detailed analysis on the equivalence of these different forms can be found in Chou & He (2002).

Remark 2.2 It should be pointed out that ‘$h^2$-uniform quadrilateral meshes’, a closely related but stronger assumption, was also used in Ewing et al. (1999) to prove some superconvergence results. The ‘$h^2$-uniformness’ requires that any two adjacent quadrilaterals form an $h^2$-parallelogram. This stronger assumption was needed to obtain a good cancellation property and then the superconvergence results (see Ewing et al., 1999, p. 782).

Remark 2.3 We may adopt a more general Assumption B that $\vartheta_Q = O(h'_Q)$ for some $\tau > 0$. Then the main difference will appear in Lemma 3.5, i.e., $|(u_h, I^*_h v_h) - (v_h, I^*_h u_h)| \leq C h^\tau \|u_h\|_0 \|v_h\|_0$, which will be used in Theorems 4.1 and 4.2. Consequently, we shall assume that $\Delta t = O(h^\tau)$ to obtain an optimal order convergence rate. Since the expected error is $O(h^2 + \Delta t^2)$, a linear rate in $h_Q$ is a practical assumption that avoids unnecessary complexities.

In order to describe the finite volume element scheme, we introduce a dual partition $\Omega^*_h$, whose elements are called control volumes. As shown in Fig. 3, each edge of $Q \in \Omega_h$ is partitioned into three segments so that the ratio of these segments is 1:4:1. We connect these partition points with line
Fig. 3. A generic quadrilateral $Q$ is partitioned into nine subregions.

segments to the corresponding points on the opposite edge. This way, each quadrilateral of $\Omega_h$ is divided into nine sub-quadrilaterals $Q_z$, with $z \in Z_h(Q)$, where $Z_h(Q)$ is the set of the vertices, the midpoints of the edges and the centre of $Q$. For each node $z \in Z_h = \bigcup_{Q \in \Omega_h} Z_h(Q)$ we associate a control volume $V_z$, which is the union of the subregions $Q_z$ containing the node $z$. Therefore we obtain a collection of control volumes covering the domain $\Omega$. This is the dual partition $\Omega^*_h$ of the primal partition $\Omega_h$. We denote the set of interior nodes of $Z_h$ by $Z^0_h$.

Remark 2.4 The dual partition introduced here is different to the one used in Yang (2006), where the sub-quadrilaterals were constructed based on the partition ratio 1:2:1. This new dual partition allows us to control nonsymmetry of the bilinear form $(\cdot, I^*_h \cdot)$ (see Lemma 3.5) and ensure coercivity of the bilinear form $a_h(\cdot, I^*_h \cdot)$ (see Lemma 3.8). Both forms will be defined later.

2.2 Finite volume element scheme

Now we formulate the finite volume element method for the model problem (1.1). Given an interpolation node $z \in Z^0_h$, integrating the first equation in (1.1) over the associated control volume $V_z$ and applying Green’s formula, we obtain

$$\int_{V_z} u_t \, dx - \int_{\partial V_z} a \nabla u \cdot n \, ds = \int_{V_z} f(x, t) \, dx, \quad (2.4)$$

where $n$ denotes the unit outer normal vector on $\partial V_z$. The above formulation also states that we have an integral conservation form on the control volume.

The integral form (2.4) can be further written in a variational form similar to the finite-element method, with the help of a transfer operator $I^*_h: S_h \to S^*_h$ from the trial space to the test space defined in Chou & Li (2000) by

$$I^*_h v = \sum_{z \in Z^0_h} v(z) \Psi_z, \quad (2.5)$$

where

$$S^*_h = \left\{ v \in L^2(\Omega) : v|_{V_z} \text{ is constant } \forall z \in Z^0_h ; v|_{V_z} = 0 \quad \forall z \in \partial \Omega \right\} \quad (2.6)$$

and $\Psi_z$ is the characteristic function of the control volume $V_z$. 

We multiply (2.4) by \( v_h(z) \) and sum over all \( z \in \mathcal{Z}_h \) to obtain
\[
(u_t, I_h^* v_h) + a_h(u, I_h^* v_h) = (f, I_h^* v_h) \quad \forall v_h \in S_h,
\] (2.7)
where the bilinear form \( a_h(\cdot, I_h^* \cdot) \) is defined as follows. For any \( u \in H^1_0(\Omega) \) and \( v_h \in S_h \) we have
\[
a_h(u, I_h^* v_h) = -\sum_{z \in \mathcal{Z}_h} v_h(z) \int_{\partial V_z} a \nabla u \cdot \mathbf{n} \, ds.
\] (2.8)

Let \( N \) be a positive integer. For simplicity of presentation, we consider a uniform time step \( \Delta t = T/N \) and set \( t_n = n \Delta t \quad (0 \leq n \leq N) \). Let
\[
u^n = u(\cdot, t_n), \quad \tilde{u}^n = u^n - u^{n-1} / \Delta t, \quad \tilde{D}u^n = \left( \frac{3}{2} u^n - 2 u^{n-1} + \frac{1}{2} u^{n-2} \right) / \Delta t.
\]
A fully discrete finite-volume scheme for (1.1) is formulated as follows. Find \( u^n_h \in S_h \quad (2 \leq n \leq N) \) such that, for any \( v_h \in S_h \), we have
\[
(\tilde{D}u^n_h, I_h^* v_h) + a_h(u^n_h, I_h^* v_h) = (f(t_n), I_h^* v_h),
\] (2.9)
with the initial approximations \( u^0_h \) and \( u^1_h \) given by
\[
u^0_h = R_h u_0,
\] (2.10)
\[
(\tilde{u}^1_h, I_h^* v_h) + a_h(u^1_h, I_h^* v_h) = (f(t_1), I_h^* v_h) \quad \forall v_h \in S_h,
\] (2.11)
where \( R_h: H^1_0(\Omega) \cap H^3(\Omega) \to S_h \) is the elliptic (Ritz) projection defined by
\[
a_h(R_h u, I_h^* v_h) = a_h(u, I_h^* v_h) \quad \forall v_h \in S_h.
\] (2.12)

**Remark 2.5** The scheme (2.9) can be written as
\[
\frac{3}{2} (u^n_h, I_h^* v_h) + \Delta t a_h(u^n_h, I_h^* v_h) = 2(u^{n-1}_h, I_h^* v_h) - \frac{1}{2} (u^{n-2}_h, I_h^* v_h) + \Delta t (f(t_n), I_h^* v_h), \quad n \geq 2.
\]
Set \( A(\cdot, \cdot) = \frac{3}{2} (\cdot, I_h^* \cdot) + \Delta t a_h(\cdot, I_h^* \cdot) \). From Lemmas 3.4, 3.6 and 3.8, we know that the bilinear form \( A(\cdot, \cdot) \) is bounded and coercive on \( S_h \) for sufficiently small \( h \). The right-hand side of the above equation defines a continuous linear functional on \( S_h \), for given \( u^{n-1}_h, u^{n-2}_h \) and \( f(t_n) \). Therefore there exists a unique solution \( u^n_h \) at each time step by the Lax–Milgram theorem (see Brenner & Scott, 2002).

**3. Properties of quadrilateral meshes and the bilinear forms**

In this section we prove some lemmas regarding properties of quadrilateral meshes and the bilinear forms defined in Section 2. Let \( P_1 \) and \( P_2 \) be two points. We use \( P_1 P_2 \) to denote the line segment, \( |P_1 P_2| \) to denote its length and \( \overrightarrow{P_1 P_2} \) to denote the vector from point \( P_1 \) to point \( P_2 \).

According to Lemmas 3.1–3.4 in Yang (2006), we have the following results.

**Lemma 3.1** Let \( Q = P_1 P_2 P_3 P_4 \in \Omega_h \). Mesh assumption B is equivalent to the following condition:
\[
|\overrightarrow{P_1 P_2} + \overrightarrow{P_3 P_4}| = O(h_Q^2).
\] (3.1)
LEMMA 3.2 Suppose that \( \Omega_h \) satisfies Mesh assumption B, \( Q = P_1 P_2 P_3 P_4 \in \Omega_h \) and
\[
\frac{|P_2 P_5|}{|P_2 P_4|} = \frac{|P_3 P_6|}{|P_3 P_4|} = d,
\]
where \( 0 \leq d \leq 1 \) is an arbitrary constant. Then we have
\[
|P_6 P_5 + P_1 P_4| = O(h^2_Q). \tag{3.2}
\]

LEMMA 3.3 Under the same assumptions as in Lemma 3.2, we have
\[
|\angle P_5 P_6 P_4 - \angle P_4 P_1 P_5| = O(h_Q), \tag{3.3}
\]
\[
\angle P_5 P_6 P_4 + \angle P_6 P_4 P_1 = \pi + O(h_Q). \tag{3.4}
\]

It is assumed that there exist a pair of integers \( n_x \) and \( n_y \) such that the cardinality of \( \Omega_h \) is equal to \( n_x n_y \), and we can assign each \( Q \in \Omega_h \) a pair of integers \( (i, j) \), where \( 0 \leq i \leq n_x - 1 \) and \( 0 \leq j \leq n_y - 1 \). Thus we label \( Q \) with the subscript \( (i, j) \) and denote its vertices by \( x_i, x_{i+1}, x_i + j, x_{i+1} + j, x_{i+j} \) and \( x_{i+j+1} \), corresponding to \( P_1, P_2, P_3 \) and \( P_4 \) in Fig. 2. Let \( v_i, v_j = 0 \) or \( \frac{1}{2} \). Then the midpoints of the edges of \( Q \) are denoted by \( x_{i+v_i + 1/2, j + v_j + 1/2} \), where \( v_i + v_j = \frac{1}{2} \), and the centre of \( Q \) is denoted by \( x_{i+1/2, j+1/2} \).

We now define some discrete norms on \( S_h \). For any \( u_h \in S_h \) we have
\[
\| u_h \|_0^2 = (v_h, I_h^* v_h), \quad \| u_h \|_{0,h}^2 = (I_h^* v_h, I_h^* v_h), \tag{3.5}
\]
\[
\| u_h \|_{1,h}^2 = \sum_{Q \in \Omega_h} \| u_h \|_{1,h,Q}^2 = \sum_{Q \in \Omega_h} \left( \sum_{v_i = 0, \frac{1}{2}, 1} \sum_{v_j = 0, \frac{1}{2}, 1} \left( \delta_x u_h \left( x_{i+v_i, j+v_j} \right) \right)^2 \right), \tag{3.6}
\]
where
\[
\delta_x u_h \left( x_{i+v_i, j+v_j} \right) = u_h \left( x_{i+v_i, j+v_j} \right) - u_h \left( x_{i+v_i+1/2, j+v_j} \right),
\]
\[
\delta_y u_h \left( x_{i+v_i, j+v_j} \right) = u_h \left( x_{i+v_i, j+v_j} \right) - u_h \left( x_{i+v_i, j+v_j+1/2} \right).
\]

The following lemma indicates that the discrete norms are equivalent to the corresponding \( L^2 \)-norm or \( H^1 \)-seminorm.

LEMMA 3.4 Assume that \( \Omega_h \) satisfies Mesh assumption A. Then there exist positive constants \( C_0 \) and \( C_1 \), independent of \( h \), such that, for any \( u_h \in S_h \), we have
\[
C_0 \| u_h \|_0 \leq \| u_h \|_0 \leq C_1 \| u_h \|_0, \tag{3.7}
\]
\[
C_0 \| u_h \|_0 \leq \| u_h \|_{0,h} \leq C_1 \| u_h \|_0, \tag{3.8}
\]
\[
C_0 \| u_h \|_1 \leq \| u_h \|_{1,h} \leq C_1 \| u_h \|_1. \tag{3.9}
\]
Proof. Since the partition is regular, we have

\[ \|\hat{u}_h\|_{L^2(\hat{Q})} \leq \left\| J_{FQ}^{-1} \right\|_{L^\infty(\hat{Q})}^{1/2} \|u_h\|_{L^2(Q)}, \]

\[ \|u_h\|_{L^2(Q)} \leq \left\| J_{FQ} \right\|_{L^\infty(\hat{Q})}^{1/2} \|\hat{u}_h\|_{L^2(\hat{Q})}, \]

where

\[ \left\| J_{FQ}^{-1} \right\|_{L^\infty(\hat{Q})} \leq C_{\hat{Q}}^{-2}, \quad \left\| J_{FQ} \right\|_{L^\infty(\hat{Q})} \leq C_{\hat{Q}}. \]

Therefore

\[ C_{\hat{Q}}^{-1} \|\hat{u}_h\|_{L^2(\hat{Q})} \leq \|u_h\|_{L^2(Q)} \leq C_{\hat{Q}} \|\hat{u}_h\|_{L^2(\hat{Q})}. \] (3.10)

For \( \int_Q u_h I_h^* u_h \, dx \) we have a similar estimate as follows:

\[ C_{\hat{Q}}^{-1} h_{\hat{Q}}^2 \int_{\hat{Q}} \hat{u}_h^2 I_{\hat{h}}^* \hat{u}_h \, d\hat{x} \leq \int_Q u_h I_h^* u_h \, dx \leq C_{\hat{Q}} h_{\hat{Q}}^2 \int_Q \hat{u}_h^2 I_h^* \hat{u}_h \, d\hat{x}. \] (3.11)

Let

\[ \varphi_1(x) = (x - 1)(2x - 1), \quad \varphi_2(x) = 4x(1 - x), \quad \varphi_3(x) = x(2x - 1) \]

be the local quadratic basis in one dimension. Let \( \psi_i(x) \), where \( 1 \leq i \leq 3 \), be the characteristic functions associated with the partitions \([0, 1/6], [1/6, 5/6] \) and \([5/6, 1] \), respectively. Then, using the standard tensor-product basis and the resulting interpolation form of \( \hat{u}_h \) on the reference element, we immediately obtain

\[ \|\hat{u}_h\|_{L^2(\hat{Q})}^2 = \mathbf{u}_Q (\mathcal{G}_1 \otimes \mathcal{G}_1) \mathbf{u}_Q^T, \quad \int_{\hat{Q}} \hat{u}_h I_{\hat{h}}^* \hat{u}_h \, d\hat{x} = \mathbf{u}_Q (\mathcal{G}_2 \otimes \mathcal{G}_2) \mathbf{u}_Q^T, \]

where \( \mathbf{u}_Q \in \mathbb{R}^9 \) is a vector consisting of the nodal values of \( u_h \) on \( Q \) and

\[
\mathcal{G}_1 = \left[ \int_0^1 \varphi_i \varphi_j \, dx \right]_{i,j=1}^3 = \frac{1}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},
\]

\[
\mathcal{G}_2 = \left[ \int_0^1 \varphi_i \psi_j \, dx \right]_{i,j=1}^3 = \frac{1}{648} \begin{bmatrix} 83 & 32 & -7 \\ 32 & 368 & 32 \\ -7 & 32 & 83 \end{bmatrix}.
\]

Since the matrices \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are symmetric and positive definite, it is not difficult to see that \( \|\hat{u}_h\|_{L^2(\hat{Q})}^2 \) and \( \int_{\hat{Q}} \hat{u}_h I_{\hat{h}}^* \hat{u}_h \, d\hat{x} \) are equivalent. Applying (3.10) and (3.11) and summing the result over \( \Omega_h \) yields estimate (3.7). Estimate (3.8) can be obtained similarly. Estimate (3.9) was proved in Yang (2006). \( \square \)

The bilinear form \((\cdot, I_h^*)\) is generally nonsymmetric. The next lemma measures how far it is from being symmetric.
**Lemma 3.5** Assume that $\Omega_h$ satisfies Mesh assumptions A and B. For any $u_h, v_h \in S_h$ we have

$$|(u_h, I_h^*v_h) - (v_h, I_h^*u_h)| \leq Ch\|u_h\|_0\|v_h\|_0. \tag{3.12}$$

**Proof.** For any $Q \in \Omega_h$ a change of variables in multiple integrals gives us

$$\int_Q u_h I_h^*v_h \, dx = \int_Q \hat{u}_h I_h^*v_h J_{FQ} \, d\hat{x}.$$  

A similar calculation to that in the proof of Lemma 3.4 yields

$$m(Q) \int_Q \hat{u}_h I_h^*v_h \, d\hat{x} = m(Q)\langle v_h \rangle_{(\mathcal{G}_2 \otimes \mathcal{G}_2)} u_Q^T,$$

where $m(Q)$ is the measure of $Q$. Since the matrix $\mathcal{G}_2$ is symmetric, we have

$$m(Q) \int_Q \hat{u}_h I_h^*v_h \, d\hat{x} - m(Q) \int_Q \hat{v}_h I_h^*u_h \, d\hat{x} = 0. \tag{3.13}$$

Direct calculations yield (see Fig. 2)

$$J_{FQ} = 2A_{124} + 2(A_{123} - A_{124})\hat{x} + 2(A_{134} - A_{124})\hat{y}, \quad 0 \leq \hat{x}, \hat{y} \leq 1,$$

where $A_{ijk}$ denotes the area of the triangle with vertices $P_i, P_j$ and $P_k$. It is obvious that $m(Q) = J_{FQ}(\frac{1}{2}, \frac{1}{2})$. Therefore, from Lemmas 3.1 and 3.3, we have

$$|J_{FQ} - m(Q)| \leq |A_{123} - A_{124}| + |A_{134} - A_{124}|$$

$$\leq \frac{1}{2} |P_1 P_2| ||P_2 P_3|| \sin(\angle P_1 P_2 P_3) - |P_4 P_1| \sin(\angle P_4 P_1 P_2)|$$

$$+ \frac{1}{2} |P_4 P_1| ||P_1 P_3|| \sin(\angle P_4 P_1 P_3) - |P_3 P_4| \sin(\angle P_3 P_4 P_1)|$$

$$\leq Ch^3_Q.$$

Combining the above estimates, we obtain

$$\left| \int_Q u_h I_h^*v_h \, dx - \int_Q v_h I_h^*u_h \, dx \right| \leq \left| \int_Q u_h I_h^*v_h \, dx - m(Q) \int_Q \hat{u}_h I_h^*v_h \, d\hat{x} \right|$$

$$+ \left| \int_Q v_h I_h^*u_h \, dx - m(Q) \int_Q \hat{v}_h I_h^*u_h \, d\hat{x} \right|$$

$$\leq Ch^3_Q \left( \int_Q |\hat{u}_h I_h^*v_h| \, d\hat{x} + \int_Q |\hat{v}_h I_h^*u_h| \, d\hat{x} \right).$$

By (3.10), we have

$$\int_Q |\hat{u}_h I_h^*v_h| \, d\hat{x} \leq C \|\hat{u}_h\|_{L^2(\hat{Q})} \|\hat{v}_h\|_{L^2(\hat{Q})} \leq C/\hat{h}_Q^2 \|u_h\|_{L^2(Q)} \|v_h\|_{L^2(Q)},$$
and hence
\[ \left| \int_Q u_h I_h^* v_h \, dx - \int_Q v_h I_h^* u_h \, dx \right| \leq C h Q \| u_h \|_{L^2(Q)} \| v_h \|_{L^2(Q)}. \]

Therefore
\[ |(u_h, I_h^* v_h) - (v_h, I_h^* u_h)| \leq \sum_{Q \in \Omega_h} \left| \int_Q u_h I_h^* v_h \, dx - \int_Q v_h I_h^* u_h \, dx \right| \]
\[ \leq C \sum_{Q \in \Omega_h} h Q \| u_h \|_{L^2(Q)} \| v_h \|_{L^2(Q)} \]
\[ \leq C h \| u_h \|_0 \| v_h \|_0 \]
by the Cauchy–Schwarz inequality. □

Remark 3.1 As shown in the proof of Lemma 3.5, the symmetry of the matrix \( \mathcal{G}_2 \) is needed for (3.13) to hold. The symmetry relies on the dual partition ratio 1:4:1 introduced in this paper. If the ratio of the dual segments is set as 1:2:1 then one can verify that
\[
\mathcal{G}_2 = \frac{1}{48} \begin{bmatrix}
8 & 1 & -1 \\
5 & 22 & 5 \\
-1 & 1 & 8
\end{bmatrix},
\]
which is not symmetric. Therefore
\[
m(Q) \int_Q \hat{u}_h I_h^* \hat{v}_h \, d\hat{x} - m(Q) \int_Q \hat{v}_h I_h^* \hat{u}_h \, d\hat{x} = m(Q) v_Q (\mathcal{G}_2 \otimes \mathcal{G}_2 - \mathcal{G}_2^T \otimes \mathcal{G}_2^T) u_Q^T.
\]
If the maximum eigenvalue of the matrix \( \mathcal{G}_2 \otimes \mathcal{G}_2 - \mathcal{G}_2^T \otimes \mathcal{G}_2^T \) is \( O(h Q) \) then we would obtain an almost symmetric result as in Lemma 3.5. But a simple calculation gives
\[
\mathcal{G}_2 \otimes \mathcal{G}_2 - \mathcal{G}_2^T \otimes \mathcal{G}_2^T = \frac{1}{576} \begin{bmatrix}
0 & -8 & 0 & -8 & -6 & 1 & 0 & 1 & 0 \\
8 & 0 & 8 & 0 & -22 & 0 & -1 & 0 & -1 \\
0 & -8 & 0 & 1 & -6 & -8 & 0 & 1 & 0 \\
8 & 0 & -1 & 0 & -22 & 0 & 8 & 0 & -1 \\
6 & 22 & 6 & 22 & 0 & 22 & 6 & 22 & 6 \\
-1 & 0 & 8 & 0 & -22 & 0 & -1 & 0 & 8 \\
0 & 1 & 0 & -8 & -6 & 1 & 0 & -8 & 0 \\
-1 & 0 & -1 & 0 & -22 & 0 & 8 & 0 & 8 \\
0 & 1 & 0 & 1 & -6 & -8 & 0 & -8 & 0
\end{bmatrix},
\]
which could not produce a factor \( O(h Q) \) as we expected! Then it is very difficult to measure how far the bilinear form \( (\cdot, I_h^* \cdot) \) is from being symmetric based on the 1:2:1 dual partition ratio.
Let $I_h: H^1_0(\Omega) \cap H^3(\Omega) \rightarrow S_h$ be the usual nodal interpolation operator satisfying the approximation property (see, e.g., Brenner & Scott, 2002)

$$||u - I_h u||_r \leq C h^{3-r} ||u||_3, \quad 0 \leq r \leq 2. \quad (3.14)$$

Next we consider the continuity of the bilinear form $a_h(\cdot, I_h^* \cdot)$. The proof is very similar to that for Lemma 3.6 in Yang (2006). We provide a short one for completeness.

**Lemma 3.6** There exists a constant $C > 0$, independent of $h$, such that

$$|a_h(u_h, I_h^* v_h)| \leq C \|u_h\|_1 \|v_h\|_1 \quad \forall u_h, v_h \in S_h, \quad (3.15)$$

$$|a_h(u - I_h u, I_h^* v_h)| \leq C h^2 \|u\|_3 \|v_h\|_1 \quad \forall u \in H^3(\Omega), v_h \in S_h. \quad (3.16)$$

**Proof.** Let $w$ denote $u_h$ or $u - I_h u$. In view of definition (2.8), we reorder by edges to get

$$|a_h(w, I_h^* v_h)| = \sum_{Q \in \mathcal{F}_h} \sum_{z_1, z_2 \in \mathcal{Z}_h \cap Q} |(v_h(z_1) - v_h(z_2)) \int_{\partial V_{z_1} \cap \partial V_{z_2}} a \nabla w \cdot n ds|,$$

where $z_1$ and $z_2$ are chosen in $Q$ with no repetition. It is obvious from (3.6) that

$$|v_h(z_1) - v_h(z_2)| \leq |v_h|_{1,h,Q}.$$

It follows from the trace inequality (see, e.g., Theorem 1.6.6 in Brenner & Scott, 2002) that

$$\left| \int_{\partial V_{z_1} \cap \partial V_{z_2}} a \nabla w \cdot n ds \right| \leq C a^{1/2}_Q \|w\|_{H^1(\partial V_{z_1} \cap \partial V_{z_2})} \leq C h^{1/2} \|w\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}^{1/2}.$$

If $w = u_h$ then we use the inverse estimate $\|u_h\|_{H^3(\Omega)} \leq Ch^{-1}_Q \|u_h\|_{H^1(\Omega)}$ (see Brenner & Scott, 2002) to get

$$\left| \int_{\partial V_{z_1} \cap \partial V_{z_2}} a \nabla u_h \cdot n ds \right| \leq C \|u_h\|_{H^1(\Omega)}.$$

If $w = u - I_h u$ then we use the approximation property (3.14) to get

$$\left| \int_{\partial V_{z_1} \cap \partial V_{z_2}} a \nabla (u - I_h u) \cdot n ds \right| \leq C h^{2}_Q \|u_h\|_{H^3(\Omega)}.$$

Combining the above estimates and using Lemma 3.4 gives

$$|a_h(u_h, I_h^* v_h)| \leq C \sum_{Q \in \mathcal{F}_h} \|u_h\|_{H^1(\Omega)} |v_h|_{1,h,Q} \leq C \|u_h\|_1 |v_h|_1$$

and

$$|a_h(u - I_h u, I_h^* v_h)| \leq C \sum_{Q \in \mathcal{F}_h} h^{2}_Q \|u\|_{H^3(\Omega)} |v_h|_{1,h,Q} \leq C h^{2} \|u\|_3 |v_h|_1,$$

as desired.
The following lemma about partitioned matrices was proved in Yang (2006) and will be used in this paper for the proof of the coercivity of $a_h(\cdot, I^*_h \cdot)$.

**Lemma 3.7** Let $\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_2^T & \mathcal{A}_1 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_2^T & \mathcal{B}_1 \end{bmatrix}$ be two partitioned matrices and let $\kappa \neq 0$ be a constant. Then the matrix $\begin{bmatrix} \mathcal{A} & \kappa \mathcal{B} \\ \kappa \mathcal{B}^T & \kappa^2 \mathcal{A} \end{bmatrix}$ is positive definite if and only if the matrices

$$
\begin{bmatrix} \mathcal{A}_1 + \frac{\mathcal{B}_1 + \mathcal{B}_1^T}{2} \\ \frac{1}{2}[(\mathcal{A}_2 + \mathcal{A}_2^T) + (\mathcal{B}_2 + \mathcal{B}_2^T)] 
\end{bmatrix}
$$

and

$$
\begin{bmatrix} \mathcal{A}_1 - \frac{\mathcal{B}_1 + \mathcal{B}_1^T}{2} \\ \frac{1}{2}[(\mathcal{A}_2 + \mathcal{A}_2^T) - (\mathcal{B}_2 + \mathcal{B}_2^T)] 
\end{bmatrix}
$$

are positive definite.

**Lemma 3.8** Assume that $\Omega_h$ satisfies Mesh assumptions A and B. There exists a constant $C_0 > 0$, independent of $h$, such that, for sufficiently small $h$, we have

$$
a_h(u_h, I^*_h u_h) \geq C_0 \|u_h\|^2_{1, \Omega} \quad \forall u_h \in S_h. \quad (3.17)
$$

**Proof.** We first study some properties of the auxiliary bilinear form

$$
\tilde{a}_h(u, I^*_h v_h) = - \sum_{z \in Z_h} v_h(z) \int_{\partial V_z} (\tilde{a} \nabla u) \cdot n \, ds \quad \forall u \in H^1_0(\Omega), v_h \in S_h, \quad (3.18)
$$

where $\tilde{a}|_Q = a(x_{i+\frac{1}{2}}, j+\frac{1}{2})$. We can rewrite (3.18) as

$$
\tilde{a}_h(u, I^*_h v_h) = \sum_{Q \in \Omega_h} \tilde{a}_{Q,h}(u, I^*_h v_h)
$$

$$
= \sum_{Q \in \Omega_h} \left( - \sum_{z \in Z_h(Q)} v_h(z) \tilde{a} \int_{\partial V_z \cap Q} \nabla u \cdot n \, ds \right). \quad (3.19)
$$

The Piola transformation maps a vector-valued function on $\hat{Q}$ to one on $Q$ by

$$
\mathbf{v} = \frac{1}{J_{F_Q}} J_{F_\hat{Q}} \hat{\mathbf{v}} \circ F^{-1}_Q.
$$

This transformation has the following well-known property (see Brezzi & Fortin, 1991):

$$
\int_e \mathbf{v} \cdot n \, ds = \int_{\hat{e}} \hat{\mathbf{v}} \cdot \hat{n} \, d\hat{s},
$$
where \( s \) and \( \hat{s} \) denote the arc lengths along the edges \( e \) and \( \hat{e} \), respectively, with \( \mathbf{n} \) and \( \hat{\mathbf{n}} \) as the unit normal vectors. For any \( u_h \in S_h \) let \( \alpha = (\mathbf{u}, Q, \mathbf{u}, Q) \) be a vector with \( \mathbf{u}, Q, \mathbf{u}, Q \in \mathbb{R}^6 \) defined by

\[
\mathbf{u}_x, Q = \left( \delta_x u_h \left( \mathbf{x}_{i+\frac{1}{2}, j} \right), \delta_x u_h \left( \mathbf{x}_{i+\frac{1}{2}, j+\frac{1}{2}} \right), \delta_x u_h \left( \mathbf{x}_{i+\frac{1}{2}, j+1} \right), \ldots, \delta_x u_h \left( \mathbf{x}_{i+1, j+1} \right) \right),
\begin{align*}
\mathbf{u}_y, Q &= \left( \delta_y u_h \left( \mathbf{x}_{i, j+\frac{1}{2}} \right), \delta_y u_h \left( \mathbf{x}_{i+\frac{1}{2}, j+\frac{1}{2}} \right), \delta_y u_h \left( \mathbf{x}_{i+1, j+\frac{1}{2}} \right), \ldots, \delta_y u_h \left( \mathbf{x}_{i+1, j+1} \right) \right).
\end{align*}
\]

Based on the Piola transformation, we integrate equation (3.19) along the edges of the reference element to obtain

\[
\frac{1}{a(x_{i+\frac{1}{2}, j+\frac{1}{2}})} \tilde{a}_{Q, h}(u_h, I_h^* u_h) = \alpha \mathcal{A} \mathbf{a}^T = \frac{1}{2} \alpha (\mathcal{A} + \mathcal{A}^T) \mathbf{a}^T,
\]

which involves the partition matrix

\[
\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_2 & \mathcal{A}_1 \end{bmatrix}.
\]

Here the matrices \((\mathcal{A}_1)_{6 \times 6}\) and \((\mathcal{A}_1')_{6 \times 6}\) represent the contraction distortion, while the matrices \((\mathcal{A}_2)_{6 \times 6}\) and \((\mathcal{A}_2')_{6 \times 6}\) characterize the rotational distortion. Let

\[
\varphi_1(x) = (x - 1)(2x - 1), \quad \varphi_2(x) = 4x(1 - x), \quad \varphi_3(x) = x(2x - 1)
\]

be the local quadratic basis in one dimension. The entries of these matrices are specified as follows:

\[
(\mathcal{A}_1)_{ij} = \begin{cases} 
\frac{7}{3} \int I_1 \frac{\varphi_j(\hat{y})}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \left| P_{12} P_{43} \right|^2, & 1 \leq i, j \leq 3, \\
-\frac{1}{3} \int I_1 \frac{\varphi_{i-1}(\hat{y})}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \left| P_{12} P_{43} \right|^2, & 1 \leq i, j - 3 \leq 3, \\
-\frac{1}{3} \int I_{i-3} \frac{\varphi_j(\hat{y})}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \left| P_{21} P_{34} \right|^2, & 1 \leq i - 3, j \leq 3, \\
\frac{7}{3} \int I_{i-3} \frac{\varphi_{i-3}(\hat{y})}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \left| P_{21} P_{34} \right|^2, & 4 \leq i, j \leq 6,
\end{cases}
\]

\[
(\mathcal{A}_2)_{ij} = \begin{cases} 
C_j \int I_1 \frac{\left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_1 \mathbf{P}_2 \cdot \mathbf{P}_{12} \mathbf{P}_{43} + \left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_2 \mathbf{P}_3 \cdot \mathbf{P}_{12} \mathbf{P}_{43}} {1 \leq i, j \leq 3,} \\
-C_j \int I_{i-3} \frac{\left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_1 \mathbf{P}_2 \cdot \mathbf{P}_{12} \mathbf{P}_{43} + \left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_4 \mathbf{P}_3 \cdot \mathbf{P}_{12} \mathbf{P}_{43}} {1 \leq i, j - 3 \leq 3,}, \\
-C_j \int I_{i-3} \frac{\left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_1 \mathbf{P}_2 \cdot \mathbf{P}_{21} \mathbf{P}_{34} + \left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_4 \mathbf{P}_3 \cdot \mathbf{P}_{21} \mathbf{P}_{34}} {1 \leq i - 3, j \leq 3,} \\
C_j \int I_{i-3} \frac{\left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_1 \mathbf{P}_2 \cdot \mathbf{P}_{21} \mathbf{P}_{34} + \left( \delta^2 \varphi \right) \left( 1 - \frac{\hat{y}}{5} \right)}{J_{FQ} \left( \frac{1}{5}, \hat{y} \right)} \frac{\mathbf{P}_4 \mathbf{P}_3 \cdot \mathbf{P}_{21} \mathbf{P}_{34}} {4 \leq i, j \leq 6,}
\end{cases}
\]

where \( C_1 = C_2 = 5/9 \) and \( C_3 = -1/9 \). The integral intervals are \( I_1 = [0, 1/6] \), \( I_2 = [1/6, 5/6] \) and \( I_3 = [5/6, 1] \). The matrices \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) take a similar form to those above.
We first investigate an auxiliary matrix $\tilde{A}$ for a parallelogram. Without loss of generality, we choose $\theta_Q = \angle P_4 P_1 P_2$ (see Fig. 4). Let $\kappa = |P_1 P_4|/|P_1 P_2|$. Note that the Jacobian determinant $J_{FQ}$ is a constant for a parallelogram. Then

$$\tilde{A} = \frac{|P_1 P_2|^2}{m(Q)} \tilde{M}, \quad \tilde{M} = \begin{bmatrix} \tilde{A}_1 & \kappa \tilde{A}_2 \\ \kappa \tilde{A}_2 & \kappa^2 \tilde{A}_1 \end{bmatrix}$$

with

$$\tilde{A}_1 = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12} & \tilde{A}_{11} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} \tilde{A}_{21} & \tilde{A}_{22} \\ \tilde{A}_{22} & \tilde{A}_{21} \end{bmatrix}$$

and

$$\tilde{A}_{11} = \begin{bmatrix} 83 & 32 & -7 \\ 32 & 368 & 32 \\ -7 & 32 & 83 \end{bmatrix}, \quad \tilde{A}_{12} = \frac{-1}{1944} \begin{bmatrix} 83 & 32 & -7 \\ 32 & 368 & 32 \\ -7 & 32 & 83 \end{bmatrix},$$

$$\tilde{A}_{21} = \frac{\cos \theta_Q}{81} \begin{bmatrix} -20 & -20 & 4 \\ -30 & -30 & 6 \\ 5 & 5 & -1 \end{bmatrix}, \quad \tilde{A}_{22} = \frac{\cos \theta_Q}{81} \begin{bmatrix} 5 & 5 & -1 \\ -30 & -30 & 6 \\ -20 & -20 & 4 \end{bmatrix}.$$
Let

\[ F(\alpha) = \frac{50}{729} - \frac{50}{729} \alpha - \frac{85}{2916} \alpha^2. \]

It is not difficult to verify that \( F(\alpha) \) is monotone decreasing for \( \alpha \in [0, 1] \) and has a root \( \alpha_0 \in (0, 1) \) (approximately 0.75667). We can choose \( \gamma < \alpha_0 \) in (2.2) so that \( F(1) > 0 \). Then the principal minors of the above matrices are all positive, and hence these matrices are positive definite. By Lemma 3.7, we know that the matrix \( \tilde{M} \) itself is positive definite as well. The minimum eigenvalue \( \lambda_{\gamma} > 0 \) of the matrix \( \tilde{M} \) depends only on the constant \( \gamma \). From Mesh assumption \( A \) we have

\[
\lambda_{\min}\left(\frac{\alpha \tilde{F} + \alpha^T\tilde{F}}{2}\right) \geq \lambda_{\gamma}\frac{|P_1P_2|^2}{m(Q)} \geq \frac{2\lambda_{\gamma}}{\sigma}. \tag{3.21}
\]

Now we consider the difference between the matrix \( \frac{\alpha \tilde{F} + \alpha^T\tilde{F}}{2} \) on a parallelogram and the matrix \( \frac{\alpha \tilde{F} + \alpha^T\tilde{F}}{2} \) on an asymptotically parallelogram. Let \( \tilde{D} = \frac{\alpha \tilde{F} + \alpha^T\tilde{F}}{2} - \frac{\alpha \tilde{F} + \alpha^T\tilde{F}}{2} \). By Lemmas 3.2 and 3.3, we have

\[
|P_{12}P_{43}| = |P_{21}P_{34}| = |P_1P_4| + O(h_Q^3),
\]

\[
|P_{14}P_{23}| = |P_{41}P_{32}| = |P_1P_2| + O(h_Q^3)
\tag{3.22}
\]

and

\[
\overrightarrow{P_1P_2} \cdot \overrightarrow{P_{12}P_{43}} = \overrightarrow{P_4P_3} \cdot \overrightarrow{P_{12}P_{43}} = \overrightarrow{P_1P_2} \cdot \overrightarrow{P_{21}P_{34}} = \overrightarrow{P_4P_3} \cdot \overrightarrow{P_{21}P_{34}}
\]

\[
= \overrightarrow{P_1P_4} \cdot \overrightarrow{P_{14}P_{23}} = \overrightarrow{P_2P_3} \cdot \overrightarrow{P_{14}P_{23}} = \overrightarrow{P_1P_4} \cdot \overrightarrow{P_{41}P_{32}} = \overrightarrow{P_2P_3} \cdot \overrightarrow{P_{41}P_{32}}
\]

\[
= |P_1P_2||P_1P_4| \cos \theta_Q \cos h_Q + O(h_Q^3).
\tag{3.23}
\]

By (2.3), (3.22) and (3.23), we can verify that, when \( h_Q \) is small enough, we have

\[
|(|\tilde{D}|_{ij}| \leq C \left(1 - \cos h_Q\right) h_Q^2 + h_Q^3 \frac{m(Q)}{m(Q)}, \quad 1 \leq i, j \leq 12.
\]

Mesh assumption \( A \) implies that \( h_Q^2/m(Q) \leq 2\sigma/\sqrt{1 - \gamma^2} \). So we have

\[
|(|\tilde{D}|_{ij}| \leq C \left(1 - \cos h_Q + h_Q\right), \quad 1 \leq i, j \leq 12,
\]

and hence

\[
\lambda_{\max}(\tilde{D}) \leq \|\tilde{D}\|_\infty \leq 12C(1 - \cos h_Q + h_Q) \leq Ch_Q \leq Ch.
\]

Combining the above results with (3.20), we obtain

\[
\bar{a}_{Q,h}(u_h, F_h^*u_h) = a \left( x_i^{1/2}, j^{1/2} \right) \alpha \frac{\alpha \tilde{F} + \alpha^T\tilde{F}}{2} \alpha^T
\]

\[
= a \left( x_i^{1/2}, j^{1/2} \right) \alpha \left[ (\alpha \tilde{F} + \alpha^T\tilde{F})/2 + \tilde{D} \right] \alpha^T.
\]
First, we derive some elementary inequalities that will be used in the proofs of the two main theorems. In this section we present a stability and convergence analysis for the finite volume element method.

4. Stability and convergence analysis

In this section we present a stability and convergence analysis for the finite volume element method. First, we derive some elementary inequalities that will be used in the proofs of the two main theorems. For any $u_h, v_h \in S_h$ one has

\[ 0 \leq \| u_h - v_h \|_0^2 = (u_h - v_h, I_h^* (u_h - v_h)) \]
\[ = \| u_h \|_0^2 + \| v_h \|_0^2 - (u_h, I_h^* v_h) - (v_h, I_h^* u_h). \] \hfill (4.1)

Together with Lemma 3.5, this implies that

\[ (u_h, I_h^* v_h) \leq \frac{1}{2} (\| u_h \|_0^2 + \| v_h \|_0^2 + (u_h, I_h^* v_h) - (v_h, I_h^* u_h)) \]
\[ \leq \frac{1}{2} (\| u_h \|_0^2 + \| v_h \|_0^2) + \frac{Ch}{2} \| u_h \|_0 \| v_h \|_0. \] \hfill (4.2)

From (4.1) we have

\[ \| u_h + v_h \|_0^2 = \| u_h \|_0^2 + \| v_h \|_0^2 + (u_h, I_h^* v_h) + (v_h, I_h^* u_h) \]
\[ \leq 2(\| u_h \|_0^2 + \| v_h \|_0^2). \] \hfill (4.3)

The following theorem indicates that the finite volume element method is stable in the $L^2$-norm.

**Theorem 4.1** Let $u_h^M$ be the numerical solution of the finite-volume scheme (2.9)–(2.11). There exists a positive constant $C$, independent of the discretization parameters, such that

\[ \| u_h^M \|_0 \leq C e^{T(h/A t^{1+1})} \left( \| u_h^0 \|_0 + A t \sum_{n=1}^{M} \| f(t_n) \|_0 \right), \quad 1 \leq M \leq N. \] \hfill (4.4)
Proof. Introducing two difference operators $\delta_i u^n_h = u^n_h - u^{n-i}_h$ for $i = 1, 2$, we rewrite $\Delta t \tilde{D} u^n_h = 2\delta_1 u^n_h - \frac{1}{2} \delta_2 u^n_h$. Note that, for $i = 1, 2$, we have
\[
2(\delta_i u^n_h, I^*_h u^n_h) = \delta_i ||u^n_h||^2_0 + \delta_i ||u^{n-i}_h||^2_0 + [(u^n_h, I^*_h u^{n-i}_h) - (u^{n-i}_h, I^*_h u^n_h)],
\]
where $\delta_i ||u^n_h||^2_0 = ||u^n_h||^2_0 - ||u^{n-i}_h||^2_0$. So we have
\[
\Delta t (\tilde{D} u^n_h, I^*_h u^n_h) = \delta_1 ||u^n_h||^2_0 - \frac{1}{4} \delta_2 ||u^n_h||^2_0 + \delta_1 ||u^{n-1}_h||^2_0 - \frac{1}{4} \delta_2 ||u^{n-1}_h||^2_0 \\
+ [(u^n_h, I^*_h u^{n-1}_h) - (u^{n-1}_h, I^*_h u^n_h)] - \frac{1}{4}[(u^n_h, I^*_h u^{n-2}_h) - (u^{n-2}_h, I^*_h u^n_h)].
\]
(4.5)

It is clear that
\[
\sum_{n=2}^{M} \left( \delta_1 ||u^n_h||^2_0 - \frac{1}{4} \delta_2 ||u^n_h||^2_0 \right) = \frac{3}{4} ||u^M_h||^2_0 - \frac{3}{4} ||u^{M-1}_h||^2_0 + \frac{3}{4} ||u^1_h||^2_0 + \frac{3}{4} ||u^0_h||^2_0.
\]
(4.6)

Since $\delta_2 u^n_h = \delta_1 u^n_h + \delta_1 u^{n-1}_h$, it follows from (4.3) that
\[
\sum_{n=2}^{M} \left( ||\delta_1 u^n_h||^2_0 - \frac{1}{4} ||\delta_2 u^n_h||^2_0 \right) \geq \frac{1}{2} \sum_{n=2}^{M} (||\delta_1 u^n_h||^2_0 - ||\delta_1 u^{n-1}_h||^2_0) \\
= \frac{1}{2} (||\delta_1 u^M_h||^2_0 - ||\delta_1 u^1_h||^2_0).
\]
(4.7)

By Lemma 3.5, we have
\[
\sum_{n=2}^{M} \left( [(u^n_h, I^*_h u^{n-1}_h) - (u^{n-1}_h, I^*_h u^n_h)] - \frac{1}{4}[(u^n_h, I^*_h u^{n-2}_h) - (u^{n-2}_h, I^*_h u^n_h)] \right) \\
\leq C \sum_{n=2}^{M} (||u^{n-1}_h||_0 + ||u^{n-2}_h||_0) ||u^n_h||_0.
\]
(4.8)

Since $||u^M_{h-1}||^2_0 \leq 2 ||u^M_h||^2_0 + 2 ||\delta_1 u^M_h||^2_0$, we have from (4.5)–(4.8) that
\[
\Delta t \sum_{n=2}^{M} (\tilde{D} u^n_h, I^*_h u^n_h) \geq \frac{3}{4} ||u^M_h||^2_0 - \frac{1}{4} ||u^{M-1}_h||^2_0 - \frac{3}{4} ||u^1_h||^2_0 + \frac{3}{4} ||u^0_h||^2_0 \\
+ \frac{1}{2} (||\delta_1 u^M_h||^2_0 - ||\delta_1 u^1_h||^2_0) - C \sum_{n=2}^{M} (||u^{n-1}_h||_0 + ||u^{n-2}_h||_0) ||u^n_h||_0 \\
\geq \frac{1}{4} ||u^M_h||^2_0 - \frac{3}{4} ||u^1_h||^2_0 + \frac{1}{4} ||u^0_h||^2_0 - \frac{1}{2} ||\delta_1 u^1_h||^2_0 \\
- C \sum_{n=2}^{M} (||u^{n-1}_h||_0 + ||u^{n-2}_h||_0) ||u^n_h||_0.
\]
(4.9)
On the other hand,
\[
A t (\tilde{\partial} u_h^1, I_h^* u_h^1) \geq \frac{1}{2} \left( \| u_h^0 \|_0^2 - \| u_h^1 \|_0^2 \right) + \frac{1}{2} \| \partial_1 u_h^1 \|_0^2 - C h \| u_h^0 \|_0 \| u_h^1 \|_0. \tag{4.10}
\]

Combining (4.9) and (4.10) yields
\[
A t (\tilde{\partial} u_h^1, I_h^* u_h^1) + A t \sum_{n=2}^M (\tilde{\partial} u_h^n, I_h^* u_h^n) \geq \frac{1}{4} \| u_h^M \|_0^2 - \frac{1}{4} \| u_h^1 \|_0^2 - \frac{1}{4} \| u_h^0 \|_0^2
\]
\[
- C h \sum_{n=1}^M \| u_h^{n-1} \|_0 \| u_h^n \|_0 - C h \sum_{n=2}^M \| u_h^{n-2} \|_0 \| u_h^n \|_0. \tag{4.11}
\]

From (2.9) and (2.11) we have
\[
A t (\tilde{\partial} u_h^1, I_h^* u_h^1) + A t \sum_{n=2}^M (\tilde{\partial} u_h^n, I_h^* u_h^n) + A t \sum_{n=1}^M a_h (u_h^n, I_h^* u_h^n)
\]
\[
\leq A t \sum_{n=1}^M \| f(t_n) \|_0 \| I_h^* u_h^n \|_0 \leq C_1 A t \sum_{n=1}^M \| f(t_n) \|_0 \| u_h^n \|_0.
\]

Applying (3.7) and (4.11), we obtain
\[
\frac{1}{4} \| u_h^M \|_0^2 + A t \sum_{n=1}^M a_h (u_h^n, I_h^* u_h^n) \leq \frac{1}{4} \| u_h^1 \|_0^2 + \frac{1}{4} \| u_h^0 \|_0^2 + C h \sum_{n=1}^M \left( \| u_h^{n-1} \|_0 + \| u_h^{n-2} \|_0 \right) \| u_h^n \|_0
\]
\[
+ C_1 A t \sum_{n=1}^M \| f(t_n) \|_0 \| u_h^n \|_0
\]
\[
\leq \frac{1}{4} \| u_h^1 \|_0^2 + \frac{1}{4} \| u_h^0 \|_0^2 + C h \sum_{n=1}^M \| u_h^{n-1} \|_0 \| u_h^n \|_0
\]
\[
+ C h \sum_{n=2}^M \| u_h^{n-2} \|_0 \| u_h^n \|_0 + C A t \sum_{n=1}^M \| f(t_n) \|_0 \| u_h^n \|_0.
\]

Let \( J \) be such that \( \| u_h^J \|_0 = \max_{1 \leq n \leq M} \| u_h^n \|_0 \). Then
\[
\| u_h^J \|_0^2 \leq \left( \| u_h^1 \|_0 + \| u_h^0 \|_0 + C h \sum_{n=1}^M \| u_h^{n-1} \|_0 + C A t \sum_{n=1}^M \| f(t_n) \|_0 \right) \| u_h^J \|_0.
\]

Therefore
\[
\| u_h^M \|_0 \leq \| u_h^J \|_0 \leq \| u_h^1 \|_0 + \| u_h^0 \|_0 + C h \sum_{n=1}^M \| u_h^{n-1} \|_0 + C A t \sum_{n=1}^M \| f(t_n) \|_0.
\]
Using (4.2), we have
\[ \|u_h^1\|_0 \leq (1 + C h)\|u_h^0\|_0 + C \Delta t \|f(t_1)\|_0. \]

Applying the discrete Gronwall inequality yields
\[ \|u_h^M\|_0 \leq C e^{M(h + \Delta t)} \left( \|u_h^0\|_0 + \Delta t \sum_{n=1}^{M} \|f(t_n)\|_0 \right) \]
\[ \leq C e^{T(h/\Delta t + 1)} \left( \|u_h^0\|_0 + \Delta t \sum_{n=1}^{M} \|f(t_n)\|_0 \right), \quad 1 \leq M \leq N. \]

The proof is completed by another use of (3.7). \[ \square \]

**Theorem 4.2** Let \( u \) be the solution of (1.1) and let \( u_h^n \) be the numerical solution of the finite-volume scheme (2.9)–(2.11). Assume that (1.2) and Mesh assumptions A and B hold. If \( \Delta t = O(h) \) then we have the following error estimate in the energy norm:
\[ \|u^M - u_h^M\|_0 + \left( \Delta t \sum_{n=0}^{M} \|u^n - u_h^n\|_1^2 \right)^{1/2} \leq C (\Delta t^2 + h^2), \quad 0 \leq M \leq N, \quad (4.12) \]
for sufficiently small \( h \) and \( \Delta t \), where the positive constant \( C \) is independent of the discretization parameters.

**Proof.** We decompose the error as \( u^n - u_h^n = \xi^n - \eta^n \), where \( \xi^n = u^n - R_h u^n \) and \( \eta^n = u^n - R_h u^n \), with \( R_h \) being the elliptic projection defined in (2.12). By (2.7), (2.9) and (2.11), we have the following error equations:
\[ (\tilde{\mathbf{D}}\xi^n, I_h^* v_h) + a_h(\xi^n, I_h^* v_h) = (\omega^n, I_h^* v_h), \quad n \geq 2, \]
\[ (\tilde{\mathbf{D}}\xi^1, I_h^* v_h) + a_h(\xi^1, I_h^* v_h) = (\omega^1, I_h^* v_h), \]
where
\[ \omega^n = \tilde{\mathbf{D}}\eta^n + u_t^n - \tilde{\mathbf{D}}u^n, \quad n \geq 2, \]
\[ \omega^1 = \tilde{\mathbf{D}}\eta^1 + u_t^1 - \tilde{\mathbf{D}}u^1. \]

By Lemma 3.8, definition (2.12) and the Galerkin orthogonality, we have
\[ C_0 \|I_h u - R_h u\|_1^2 \leq a_h(I_h u - R_h u, I_h^*(I_h u - R_h u)) \]
\[ = a_h(I_h u - u, I_h^*(I_h u - R_h u)). \]

It follows from Lemma 3.6 that
\[ \|I_h u - R_h u\|_1 \leq C h^2 \|u\|_3. \quad (4.13) \]

Using (3.14) and a triangle inequality, we obtain
\[ \|\eta^n\|_1 \leq \|u^n - I_h u^n\|_1 + \|I_h u^n - R_h u^n\|_1 \leq C h^2 \|u^n\|_3. \quad (4.14) \]
Suppose that $J$ is such that $\|\xi^J\|_0 = \max_{1 \leq n \leq M} \|\xi^n\|_0$, where $1 \leq M \leq N$. Since $\xi^0 = 0$, a similar argument as that in the proof of Theorem 4.1 leads to

$$\frac{1}{4} \|\xi^J\|_0^2 + C_0 \Delta t \sum_{n=1}^M \|\xi^n\|_1^2 \leq C h \sum_{n=1}^M \|\xi^{n-1}\|_0 \|\xi^n\|_0 + C \Delta t \sum_{n=1}^M \|\omega^n\|_0 \|\xi^J\|_0 + C \Delta t^2 \|\omega^1\|_0^2$$

$$\leq C (h + \Delta t) \sum_{n=1}^M \|\xi^n\|_0^2 + 8 C \left( \Delta t \sum_{n=1}^M \|\omega^n\|_0 \right)^2$$

$$+ \frac{1}{8} \|\xi^J\|_0^2 + C \Delta t^2 \|\omega^1\|_0^2. \quad (4.15)$$

Applying Taylor’s expansion with the remainder term in integral form, we get

$$\Delta t \sum_{n=2}^M \|\omega^n\|_0 \leq \Delta t \sum_{n=2}^M (\|\tilde{D}\eta^n\|_0 + \|u^n_t - \tilde{D}u^n\|_0)$$

$$\leq C \sum_{n=2}^M \left( \int_{t_{n-2}}^{t_n} \|\eta_t\|_0 \, dt + \Delta t^2 \int_{t_{n-2}}^{t_n} \|u_{tt}||0 \, dt \right)$$

$$\leq C \left( \int_0^T \|\eta_t\|_0 \, dt + \Delta t^2 \int_0^T \|u_{tt}\|_0 \, dt \right)$$

$$\leq C \left( h^2 \int_0^T \|u_t\|_3 \, dt + \Delta t^2 \int_0^T \|u_{tt}\|_0 \, dt \right).$$

Similarly, we have

$$\Delta t \|\omega^1\|_0 \leq \Delta t \|\tilde{D}\eta^1\|_0 + \Delta t \|u^1_t - \tilde{D}u^1\|_0$$

$$\leq C h^2 \int_0^{At} \|u_t\|_3 \, dt + \Delta t \int_0^{At} \|u_{tt}\|_0 \, dt. \quad (4.17)$$

Combining (4.15)–(4.17), we obtain

$$\|\xi^J\|_0^2 + \Delta t \sum_{n=1}^M \|\xi^n\|_1^2 \leq C (h + \Delta t) \sum_{n=1}^M \|\xi^n\|_0^2 + C_u (\Delta t^4 + h^4),$$

where $C_u = C (\|u_t\|_{L^1(0,T;H^3(\Omega))} + \|u_{tt}\|_{L^\infty(0,T;L^2(\Omega))} + \|u_{ttt}\|_{L^1(0,T;L^2(\Omega))})^2$.

To balance spatial and temporal errors it is sufficient to have $h = \mathcal{O}(\Delta t)$, which allows us to use (3.7) and the discrete Gronwall inequality to derive

$$\|\xi^M\|_0^2 + \Delta t \sum_{n=1}^M \|\xi^n\|_1^2 \leq C_0^{-2} \|\xi^J\|_0^2 + \Delta t \sum_{n=1}^M \|\xi^n\|_1^2 \leq C (\Delta t^4 + h^4), \quad M \geq 1.$$
for sufficiently small $h$ and $\Delta t$. By (4.14), we obtain
\[
\| \eta^n(t) \|_1 \leq Ch^2 \| u^n \|_3 \leq Ch^2 \left( \| u_0 \|_3 + \int_0^t \| u_r \|_3 \, dr \right).
\]

Combining the above two estimates, we get the desired estimate. □

5. Numerical experiments

In this section we present numerical experiments to illustrate the theoretical results presented in the previous sections. The domain is $\Omega = [0, \pi]^2$, the permeability coefficient $a(x, y) = (x + 1)^2 + y^2$, the exact solution $u(x, y, t) = e^{-0.1t} \sin(x) \sin(y)$ and the final time $T = 0.2$. The right-hand side $f(x, y, t)$ is computed accordingly.

We test the finite-volume method first on a family of rectangular meshes (see Fig. 5) that have $M = 4, 8, 16, 32$ or 64 partitions in both the $x$- and $y$-directions and then on a set of quadrilateral meshes generated by
\[
\begin{align*}
    x_{i,j} &= \frac{\pi}{M} i + \frac{\pi}{4M} \sin(j) \text{rand}(), \\
    y_{i,j} &= \frac{\pi}{M} j + \frac{\pi}{4M} \sin(i) \text{rand}(),
\end{align*}
\]

where $\sin(i)$ and $\sin(j)$ use the radian unit for angles and $\text{rand}()$ generates a random number in $(0, 1)$. An implementation of the random function $\text{rand}()$ depends on the computer platform used but is available in the standard library of most C/C++ compilers.

Given a spatial mesh parameter $h = \pi / M$, the time step size is chosen as $\Delta t = T / (2\pi) h$. We use the first-order backward Euler as a starter scheme (see, e.g., Thomée, 2006). The preconditioned bi-conjugate gradient stabilized method is employed to solve the nonsymmetric discrete linear systems.

![An example for asymptotically parallelogram quadrilateral meshes (used in our numerical experiments).](image)
TABLE 1 Errors and convergence rates on the rectangular meshes

<table>
<thead>
<tr>
<th>$M$</th>
<th>$|I_h u_N^N - u_h^N|_{l^\infty}$</th>
<th>Rate</th>
<th>$|I_h u_N^N - u_h^N|_{1,h}$</th>
<th>Rate</th>
<th>$|u_h^N|_E$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$8.563 \times 10^{-3}$</td>
<td>—</td>
<td>$2.202 \times 10^{-2}$</td>
<td>—</td>
<td>$2.032 \times 10^{-2}$</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>$2.074 \times 10^{-3}$</td>
<td>2.04</td>
<td>$5.546 \times 10^{-3}$</td>
<td>1.98</td>
<td>$5.226 \times 10^{-3}$</td>
<td>1.95</td>
</tr>
<tr>
<td>16</td>
<td>$5.138 \times 10^{-4}$</td>
<td>2.01</td>
<td>$1.388 \times 10^{-3}$</td>
<td>1.99</td>
<td>$1.312 \times 10^{-3}$</td>
<td>1.99</td>
</tr>
<tr>
<td>32</td>
<td>$1.281 \times 10^{-4}$</td>
<td>2.00</td>
<td>$3.471 \times 10^{-4}$</td>
<td>1.99</td>
<td>$3.281 \times 10^{-4}$</td>
<td>2.00</td>
</tr>
<tr>
<td>64</td>
<td>$3.200 \times 10^{-5}$</td>
<td>2.00</td>
<td>$8.676 \times 10^{-5}$</td>
<td>2.00</td>
<td>$8.241 \times 10^{-5}$</td>
<td>1.99</td>
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</tbody>
</table>

TABLE 2 Errors and convergence rates on the quadrilateral meshes

<table>
<thead>
<tr>
<th>$M$</th>
<th>$|I_h u_N^N - u_h^N|_{l^\infty}$</th>
<th>Rate</th>
<th>$|I_h u_N^N - u_h^N|_{1,h}$</th>
<th>Rate</th>
<th>$|u_h^N|_E$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$8.089 \times 10^{-3}$</td>
<td>—</td>
<td>$1.945 \times 10^{-2}$</td>
<td>—</td>
<td>$2.126 \times 10^{-2}$</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
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<td>$6.397 \times 10^{-3}$</td>
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<tr>
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<td>$1.596 \times 10^{-3}$</td>
<td>2.00</td>
</tr>
<tr>
<td>32</td>
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<td>1.96</td>
<td>$3.718 \times 10^{-4}$</td>
<td>1.97</td>
<td>$4.072 \times 10^{-4}$</td>
<td>1.97</td>
</tr>
<tr>
<td>64</td>
<td>$3.426 \times 10^{-5}$</td>
<td>2.03</td>
<td>$9.316 \times 10^{-5}$</td>
<td>1.99</td>
<td>$1.024 \times 10^{-4}$</td>
<td>1.99</td>
</tr>
</tbody>
</table>

The tolerance for residuals is set as $10^{-11}$ and simple diagonal preconditioning is used. The numerical order of convergence is then measured by comparing the computed errors on two successive mesh levels.

From the proof of Theorem 4.2 we know that the second-order error estimate in the energy norm is essentially determined by the second-order convergence in $\Delta t$ and $h$ of the following quantity:

$$\|u_h^N\|_E = \|I_h u_N^N - u_h^N\|_0 + \left(\frac{\Delta t}{\|I_h u_N^N - u_h^N\|_{1,h}}\right)^{1/2},$$

due to the approximation property (3.14), the norm equivalences (3.7) and (3.9) and the estimate (4.13). We thus measure $\|u_h^N\|_E$ in the numerical experiments in lieu of the energy norm defined in (4.12). The numerical results listed in Tables 1 and 2 clearly reflect the second-order convergence of $\|u_h^N\|_E$ for both the rectangular and the quadrilateral meshes, which agrees very well with our theoretical findings. As a by-product, we can also obtain second-order accuracy in the max-norm $\|I_h u_N^N - u_h^N\|_\infty$ and the discrete $H^1$-norm $\|I_h u_N^N - u_h^N\|_{1,h}.$

6. Concluding remarks

In this paper we presented a quadratic finite volume element method for parabolic problems on quadrilateral meshes. A suitable dual partition was introduced to control the nonsymmetry of the bilinear forms. A second-order convergence in the $L^2(0, T; H^1(\Omega))$-norm was derived and verified in numerical experiments. However, there are technical difficulties in deriving error estimates in $L^\infty(0, T; H^1(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$ similar to those for the finite-element methods. The main reason for this deficiency is that we are not able to derive the optimal $L^2$-norm error estimate for the Ritz projection for the bilinear form $a_h(\cdot, I_h^N \cdot).$ Such a deficiency exists on triangular and quadrilateral meshes (see Liebau, 1996; Li et al., 2000; Yang, 2006; Xu & Zou, 2009). How to obtain the optimal $L^2$-norm error estimates for finite volume element methods based on multidimensional higher-order elements is still an open problem.
The coercivity of the bilinear form $a_h(\cdot, I_h^* \cdot)$ plays a critical role in the stability and convergence analysis of the finite volume element scheme. The proof of the coercivity (Lemma 3.8) is demanding. The usual techniques used in the analysis of linear finite-volume methods do not help here. However, the coercivity is related to the positive definiteness of certain matrices at the element level. This challenging issue is then resolved by breaking up the distortion of affine mappings into contraction and rotation and employing partitioned matrices for size reduction. We hope that these techniques could be extended to the analysis of other higher-order finite-volume methods in multiple dimensions.

Quadrilateral meshes have been used in a variety of applications and the asymptotically parallelogram assumption has been widely accepted in the literature (see Süli, 1992; Ewing et al., 1999; Arnold et al., 2002; Flemisch & Wohlmuth, 2007), although it takes several different but equivalent forms. It would be very interesting to see whether the assumption can be significantly weakened.

Apparently, the quadratic finite-volume elements can be combined with the backward Euler temporal discretization. Even though the implicit method is unconditionally stable, the condition $\Delta t = O(h^2)$ is still needed to balance the spatial and temporal errors. This is prohibitive for fine spatial meshes. With the two-step implicit scheme presented in this paper, the time step size is only $\Delta t = O(h)$, but a second-order convergence is obtained in numerical solutions with reduced computational costs.

The quadratic finite-volume method can also be applied to convection–diffusion equations, in conjunction with treatments of convection by some upwinding techniques or the characteristic methods. The development of finite-volume elements of order higher than two in multiple dimensions and their analysis are even more challenging. These are currently under investigation and will be reported in our future work.

REFERENCES


