COLOSTATE SPRING 2017 MATH 340 EXAM 2

Thu. 04/13/2017

NAME: Answers CSUID#: 100/100

SECTION: All 8 sections

Problem	Score
1	10
2	10
3	10
4	10
5	15
6	15
7	15
8	15
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use one letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer
(2 points for each item, no partial credit).

(i) (T) (F) The following two vector functions

$$\mathbf{u}_1(t) = \begin{bmatrix} e^t\cos(t) \\ e^t(\cos(t) - \sin(t)) \end{bmatrix}, \qquad \mathbf{u}_2(t) = \begin{bmatrix} e^t\sin(t) \\ e^t(\cos(t) + \sin(t)) \end{bmatrix}$$

form a fundamental set of solutions for the linear ODE system $\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}\mathbf{x}(t)$.

(ii) (T) (F) All the solutions of the ODE $4x'' + 4x' + 17x = 0$ are in the form $x(t) = e^{-t}(C_1\cos(2t) + C_2\sin(2t))$, where C_1, C_2 are arbitrary constants.

(iii) (T) (F) The ODE $4x''+4x'+17x = 0$ describes an underdamped harmonic motion.

(iv) (T) (F) For the square matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{v}_1 = [1, 0, 0]^T$ is an eigenvector associated with $\lambda = 2$.

(v) (T) (F) A 3×3 real matrix has at least one real eigenvalue.

(10 points) *Problem 2.* Consider the autonomous ODE system

$$\begin{cases} x'(t) = f_1(x, y) = -2y + x(x^2 + 4y^2 - 4) \\ y'(t) = f_2(x, y) = \frac{1}{2}x - y(x^2 + 4y^2 - 4) \end{cases}$$

Determine whether the following statements are true or false.
Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) $x(t) = 2\cos(t), y(t) = \sin(t)$ represents a solution curve.

(ii) (T) (F) There is a unique solution curve that passes through point $(2, 0)$.

(iii) (T) (F) $\dfrac{\partial f_1}{\partial x}$ is continuous on the whole xy-plane.

(iv) (T) (F) There exists a solution curve that passes through point $(1, 0)$.

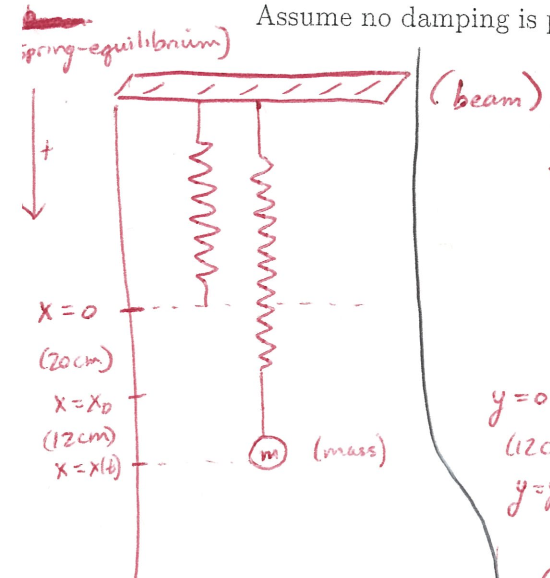
(v) (T) (F) There is a solution curve passing through both points $(1, 0)$ and $(3, 0)$.

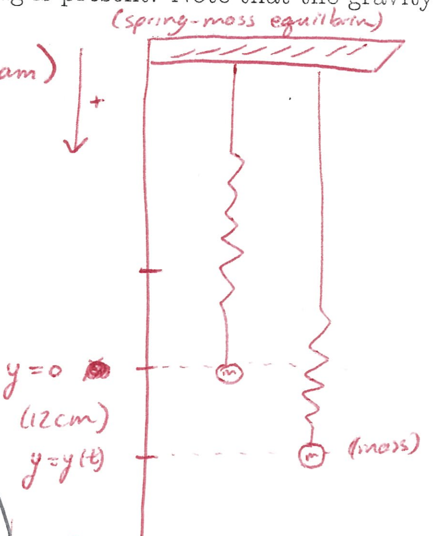
(10 points) *Problem 3.* In an experiment, a 3-kg mass is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 20 cm. Then the mass is displaced 12 cm downward from its spring-mass equilibrium and released from rest. Set up (but **DO NOT** solve) an initial value problem that models the motion.

(i) Write down a second order ODE for the displacement of the mass.

(ii) Write down two initial conditions.

Assume no damping is present. Note that the gravity acceleration constant is $g = 9.8m/s^2$.





$\frac{5}{2}$
$\overline{3}$

①

$$\begin{cases} mx'' + \mu x' + kx = mg \\ x'' + \frac{\mu}{m}x' + \frac{k}{m}x = g \end{cases}$$

Using ①: $m = 3kg$ (1point)

1point $x_0 = 20cm = 0.2m$

1point $\mu = 0$

$k = \frac{mg}{x_0} = \frac{(3)(9.8)}{(0.2)} = 147$ (2points)

Forms:

$$\begin{cases} 3x'' + \frac{(3)(9.8)}{(0.2)}x = (3)(9.8) \\ x'' + \frac{9.8}{0.2}x = 9.8 \quad \text{(2points)} \\ x'' + 49x = 9.8 \end{cases}$$

$3x'' + 147x = 29.4$

Initial Condition: (2pts)

$x(0) = 20cm + 12cm = 32cm = \underline{0.32m}$

$x'(0) = 0$ (1pts)

②

$$\begin{cases} my'' + \mu y' + ky = 0 \\ y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0 \end{cases}$$

Using ②: $m = 3kg$ (1pt)

$x_0 = 20cm = 0.2m$ (1pt)

$\mu = 0$ (1pt)

$k = \frac{mg}{x_0} = \frac{(3)(9.8)}{(0.2)} = 147$ (2pts)

Forms:

$$\begin{cases} 3y'' + 147y = 0 \\ 3y'' + \frac{(3)(9.8)}{(0.2)}y = 0 \quad \text{(2pts)} \\ y'' + \frac{9.8}{0.2}y = 0 \end{cases}$$

$y'' + 49y = 0$

Initial Condition:

$y(0) = 12cm = \underline{0.12m}$ (2pts)

$y'(0) = \underline{0}$ (1pts)

(10 points) *Problem 4.* Consider a second order ODE

$$x''(t) + 2x'(t) - 3x(t) = x(t)^3.$$

(i) Convert the above ODE into a first-order ODE system by setting $x_1 = x, x_2 = x'$.

(ii) Find all equilibria of the first-order ODE system.

5 (i)

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -2x_2 + 3x_1 + x_1^3 \end{pmatrix}}$$

5 (ii)

$$\begin{pmatrix} x_2 \\ -2x_2 + 3x_1 + x_1^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0$$

$$3x_1 + x_1^3 = 0$$

$$x_1(3 + x_1^2) = 0$$

$$\Rightarrow \quad x_1 = 0, \quad x_1 = \pm i\sqrt{3}$$

Eqm pts

$$\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} i\sqrt{3} \\ 0 \end{pmatrix}, \begin{pmatrix} -i\sqrt{3} \\ 0 \end{pmatrix}}$$

OR

$$\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$ (only real Equ pts)

Both ok.

(15 points) *Problem 5.* Consider a 2nd order linear ODE $y'' - 4y' - 5y = 0$.

(i) Find a general solution;

(ii) Find the particular solution $y_p(t)$ satisfying the initial conditions $y(1) = -1, y'(1) = 0$.

(iii) Is $t = 1$ a local minimum of the function $y_p(t)$?

(i) $\lambda^2 - 4\lambda - 5 = 0$

$(\lambda-5)(\lambda+1) = 0$

$\lambda = 5 \quad \lambda = -1$

$\boxed{y(t) = c_1 e^{5t} + c_2 e^{-t}}$

(ii) $y'(t) = 5c_1 e^{5t} - c_2 e^{-t}$

$y(1) = c_1 e^5 + c_2 e^{-1} = -1$

$y'(1) = 5c_1 e^5 - c_2 e^{-1} = 0$

$c_2 = 5c_1 e^6$

$c_1 = \dfrac{-1 - c_2 e^{-1}}{e^5} = \dfrac{-1-(5c_1 e^6)e^{-1}}{e^5} = \dfrac{-1-5c_1 e^5}{e^5}$

$c_1 e^5 = -1 - 5c_1 e^5$

$6c_1 e^5 = -1$

$\boxed{c_1 = \dfrac{-1}{6e^5}}$ $\boxed{c_2 = 5\left(\dfrac{-1}{6e^5}\right)e^6 = \dfrac{-5}{6}e}$

$\boxed{y_p(t) = \dfrac{-1}{6e^5} e^{5t} - \dfrac{5}{6} e e^{-t} = \dfrac{-1}{6} e^{5t-5} - \dfrac{5}{6} e^{1-t}}$

(iii) $y''(t) = 4y'(t) + 5y(t)$

$y''(1) = 4y'(1) + 5y(1)$

$= 4(0) + 5(-1)$

$= -5 < 0$

By 2nd derivative test, $t=1$ is a local maximum, not a local minimum.

(15 points) *Problem 6.* Consider the given 4th order ODE:

$$y^{(4)}(t) - 2y''(t) + y(t) = 0.$$

(i) If $x_1(t)$ and $x_2(t)$ are both solutions of the ODE, then is $x_1(t) + x_2(t) + 20^{17}$ also a solution of the ODE?

(ii) Write down the characteristic equation and find all eigenvalues.

(iii) Find a general solution.

(i) Answer: No (2 pts correct answer)

- $(x_1+x_2+20^{17})^{(4)} - 2(x_1+x_2+20^{17})'' + (x_1+x_2+20^{17}) = 20^{17} \neq 0$.
- $x_1+x_2+20^{17}$ is not a linear combination of x_1 and x_2.

(1 pt if wrong answer with some justification about linearity or checking the "solution")

(ii) $p(\lambda) = \lambda^4 - 2\lambda^2 + 1 = 0$ (2)

2pts Equation
4pts solving
2pts each λ

$\Leftrightarrow (\lambda^2-1)^2 = 0$ (2) $\Leftrightarrow \lambda^2 = 1$ (1) $\Leftrightarrow \lambda = \pm 1$ (2) (each of multiplicity 2)

(iii)
$$\boxed{y(t) = C_1 e^{t} + C_2 t e^{t} + C_3 t e^{-t} + C_4 e^{-t}}$$

4 1 pt each particular solution

1 1 pt general solution

(15 points) *Problem 7.* Consider an ODE system $\mathbf{x}'(t) = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$.

(i) Find its **real-valued** general solution;

(ii) Classify the equilibrium at the origin (stable/unstable? center/nodal/saddle/spiral?)

(i) $A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}$

characteristic equation: $\lambda^2 - 2\lambda + 2 = 0$ (+1)

$\Rightarrow \lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = 1 \pm i \Rightarrow \lambda_1 = 1+i,\ \lambda_2 = \bar{\lambda}_1 = 1-i$ (+2)

$(A - \lambda_1 I)\vec{w} = (A - (1+i)I)\vec{w} = \begin{bmatrix} -(1+i) & 1 \\ -2 & 2-(1+i) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -(1+i)w_1 + w_2 = 0 \Rightarrow (1+i)w_1 = w_2 \Rightarrow w_1 = \frac{w_2}{1+i}$.

If $w_2 = 1+i \Rightarrow w_1 = 1$. Then $\vec{w} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{v}_1 + i\vec{v}_2$. (+4)

Then, the real-valued solution is

$\vec{x}(t) = c_1 e^t \left[\cos(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + c_2 e^t \left[\sin(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

$\Rightarrow \vec{x}(t) = c_1 e^t \begin{pmatrix} \cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(t) \\ \sin(t) + \cos(t) \end{pmatrix}$ (+4)

(ii) real part of λ_1, λ_2 is 1, positive $\Rightarrow$ unstable (+2)

the eigenvalues are complex $\Rightarrow$ spiral (+2)

$$A-\lambda I = \begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}, \quad (A-\lambda I)^2 = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \quad (A-\lambda I)^3 = 0$$

(15 points) *Problem 8.* Consider a linear ODE system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$.

Find a general solution.

2 pts

$$\boxed{\lambda_1 = 1,} \quad \boxed{\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}}$$ 2 pts

$$\left(\begin{array}{ccc|c}0&1&1&0\\0&0&1&0\\0&0&0&0\end{array}\right) \Rightarrow \vec{v}_1 = r\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$

$$\left(\begin{array}{ccc|c}0&1&1&1\\0&0&1&0\\0&0&0&0\end{array}\right) \to \begin{pmatrix}0&1&0&1\\0&0&1&0\\0&0&0&0\end{pmatrix} \Rightarrow \boxed{\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}} + r\begin{pmatrix}1\\0\\0\end{pmatrix}$$

+1 pt for two more ~~linearly ind.~~ vectors

+3 pts for linear ind.

OR

+2 for only one other linearly ind. vector.

$$(A-\lambda I)\vec{v}_3 = \vec{v}_2$$

$$\left(\begin{array}{ccc|c}0&1&1&0\\0&0&1&1\\0&0&0&0\end{array}\right) \to \left(\begin{array}{ccc|c}0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right) \Rightarrow \boxed{\vec{v}_3 = \begin{pmatrix}0\\-1\\1\end{pmatrix}} + r\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = c_1 e^t\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 e^t\left(\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\0\end{pmatrix}\right) + c_3 e^t\left(\begin{pmatrix}0\\-1\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$$

$$\underbrace{c_1 e^t\begin{pmatrix}1\\0\\0\end{pmatrix}}_{1} + \underbrace{c_2 e^t\begin{pmatrix}t\\1\\0\end{pmatrix}}_{2} + \underbrace{c_3 e^t\begin{pmatrix}t^2/2+t\\t\\1\end{pmatrix}}_{3}$$

- 1pt per term (6 total)
- 1pt for constants