ColoState Spring 2017 Math 340 Exam 1

Thu. 03/02/2017

NAME: Answers CSUID: 100/100

SECTION: All 8 Sections

Problem	Score
1	10
2	10
3	10
4	10
5	15
6	15
7	15
8	15
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use one letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The constant function $x = -1$ is a solution to the ODE $x'(t) = (x-1)\ln(x+2)$.

(ii) (T) (F) The ODE $(x + 2y^2)dx + 2xydy = 0$ is exact.

(iii) (T) (F) For a given logistic population model $\dfrac{dP(t)}{dt} = \left(1 - \dfrac{P}{K}\right)P$, the rate at which the population is increasing is at its greatest when the population is at one-half of its carrying capacity K.

(iv) (T) (F) If $y(t)$ is a solution of the ODE $\cos(t)x'(t) - \sin(t)x(t) = 0$, then $\sqrt{2}\, y(t)$ is also a solution of the ODE.

(v) (T) (F) If $y_1(t), y_2(t)$ are both solutions of the ODE

$$\cos(t)x'(t) - \sin(t)x(t) = e^t,$$

then $y_1(t) - y_2(t)$ is a solution of the ODE

$$\cos(t)x'(t) - \sin(t)x(t) = 0.$$

(10 points) *Problem 2.* Consider an initial value problem (IVP)

$$\begin{cases} x'(t) = f(t,x) = \tan(x)/(1+t^2), \\ x(1) = \pi/4. \end{cases}$$

Determine whether the following statements are true or false. Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) $f(t,x)$ is continuous on the entire tx-plane.

(ii) (T) (F) There exists a rectangle containing the point $(1, \pi/4)$ in which $f(t,x)$ is continuous.

(iii) (T) (F) There exists a rectangle containing the point $(1, \pi/4)$ in which $\frac{\partial f}{\partial x}(t,x)$ is positive.

(iv) (T) (F) The IVP has a unique solution.

(v) (T) (F) If we change the initial condition to $x(1) = \pi/2$, then the new IVP still has a unique solution.

(10 points) *Problem 3.* An object with mass $m = 1kg$ is released from rest at a height 10 meter above the ground. Assume the air resistance can be ignored.

(i) Establish an ODE for the height $y(t)$ of the object.

(ii) Write down initial conditions for the object's motion.

(iii) How long will it take for the object to reach the ground?

Note: The gravitational constant $g = 9.8m/s^2$.

3 pts i) $\frac{d^2y}{dt^2} = \frac{dv}{dt} = -9.8$ 1 pt for $\int a = v$, $\int v = y$

2 pts ii) $y(0) = 10$, $v(0) = 0$ 1 pt each

5 pts iii) $\frac{dv}{dt} = -9.8 \Rightarrow v(t) = -9.8t + C$ 1 pt

$v(0) = 0 \Rightarrow C = 0$

$\frac{dy}{dt} = -9.8t \Rightarrow y(t) = \frac{-9.8t^2}{2} + C$

$y(0) = 10 \Rightarrow C = 10$ 1 pt

$y(t) = \frac{-9.8t^2}{2} + 10$

Solve $y(t) = 0$ for t:

$\frac{-9.8t^2}{2} + 10 = 0 \Rightarrow t^2 = 10 \cdot \frac{2}{9.8}$ 1 pt

$\Rightarrow t = \sqrt{\frac{20}{9.8}} \approx \sqrt{2}$

$= \sqrt{\frac{10}{4.9}}$ 2 pts

$= \frac{\sqrt{40 \cdot 4.9}}{9.8} \approx \sqrt{2.040816}$

$= \frac{\sqrt{196}}{9.8}$ ≈ 1.42857

$= \frac{14}{9.8}$

(10 points) *Problem 4.* Find the solution of $x'(t) = \cos^2(x)/(1+t^2)$ that satisfies $x(0) = \frac{\pi}{4}$.

General Solution

$(\star)$: $\dfrac{dx}{dt} = \dfrac{\cos^2 x}{1+t^2} \rightarrow \dfrac{dx}{\cos^2 x} = \dfrac{dt}{1+t^2}$ (2pts for seperating variables)

$$\int \sec^2 x \, dx = \int \frac{dt}{1+t^2}$$

(1pt / integral)
(1pt for + C)

$$\tan x(t) = \arctan(t) + C$$

2pts

1pt. $\underbrace{\arctan(\tan(x(t)))} = \arctan(\arctan(t) + C)$ (2pts for solving for $x(t)$)

1pt. $\boxed{x(t) = \arctan(\arctan t + C)}$

Initial Condition

$(\star)$: $\dfrac{\pi}{4} = \arctan(\arctan(0) + C)$ (2pts for algebra)

$$\tan\left(\frac{\pi}{4}\right) = \tan(\arctan(\arctan(0) + C))$$

$$1 = \underbrace{\arctan(0)} + C$$

$$1 = 0 + C \Rightarrow \underline{C = 1}$$

$\boxed{x(t) = \arctan(\arctan t + 1)}$ (1pt for writing down $x(t)$)

Check: $x(0) = \arctan(\arctan(0) + 1) = \arctan(1) = \dfrac{\pi}{4}$.

General Solution: 7 points
Initial Condition: 3 points

(15 points) *Problem 5.* Given a linear ODE: $\cos(t)x'(t) - \sin(t)x(t) = e^t$,

(i) Find a general solution.

(ii) Find the specific solution satisfying $x(0) = 2017$.

(i) $\cos(t)x' - \sin(t)x = e^t$

$$\Rightarrow \quad x' - \frac{\sin(t)}{\cos(t)}x = \frac{e^t}{\cos(t)} \qquad \text{1 pt}$$

Integrating factor: $u(t) = e^{-\int \frac{\sin(t)}{\cos(t)}dt} = e^{\ln(\cos(t))} = \cos(t)$ 3pts

Multiply both ~~sides~~ sides: $\cos(t)\left(x' - \frac{\sin(t)}{\cos(t)}x\right) = \frac{e^t}{\cancel{\cos(t)}} \cdot \cancel{\cos(t)}$

$$\Rightarrow \quad (\cos(t) \cdot x)' = e^t$$

$$\Rightarrow \quad \cos(t)x = \int e^t dt$$

$$\Rightarrow \quad \cos(t)x = e^t + C$$

6 pts

$$\Rightarrow \quad \boxed{x(t) = \frac{e^t + C}{\cos(t)}}$$ General Solution. 1 pt

(ii) $x(0) = 2017 \Leftrightarrow \frac{e^0 + C}{\cos(0)} = 2017$

$$\Leftrightarrow \quad C = 2016$$

3 pts

$$\boxed{x(t) = \frac{e^t + 2016}{\cos(t)}}$$ Particular Solution. 1 pt

(15 points) *Problem 6.* It is known that the ODE

$$Pdx + Qdy = (xy - 2)dx + (x^2 - xy)dy = 0$$

is not exact but has an integrating factor that depends only on x.

(i) Find such an integrating factor.

(ii) Find a general solution of the ODE.

See Textbook P.72 Ex. 6.39

(i) $\frac{\partial P}{\partial y} = x$, $\frac{\partial Q}{\partial x} = 2x - y$, $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x^2 - xy}(-x+y) = \frac{-1}{x}$ is a function of x only

3pts

$\mu(x) = e^{\int h(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \boxed{\frac{1}{x}}$ is a desired integrating factor

2pts

(ii) Multiplying the original ODE through by $\frac{1}{x}$ to get

$$\left(y - \frac{2}{x}\right)dx + (x - y)dy = Mdx + Ndy = 0$$

this is indeed an exact ODE.

There exists $F(x,y)$ such that $\begin{cases} \frac{\partial F}{\partial x} = M(x,y) = y - \frac{2}{x} \\ \frac{\partial F}{\partial y} = N(x,y) = x - y \end{cases}$ 4pts

Integrating the 1st one to get

2pts $F(x,y) = xy - 2\ln|x| + \varphi(y)$

Then differentiating to produce

2pts $\frac{\partial F}{\partial y} = x + \varphi'(y)$

compare to get

$\varphi'(y) = -y$

$\varphi(y) = -\frac{1}{2}y^2$

So $F(x,y) = xy - 2\ln|x| - \frac{y^2}{2}$

A general solution is $\boxed{xy - 2\ln|x| - \frac{y^2}{2} = C}$ arbitrary constant

2pts

(15 points) *Problem 7.* Given an ODE $x'(t) = (x-1)\ln(x+2)$,

9 pts (i) Find all equilibrium solutions and determine the stability of each.

6 pts (ii) Sketch three representative solution curves on the tx-plane.
Requirement: They should not be straight lines.

a. $0 = (x-1)\ln(x+2)$

$\boxed{x=1, \quad x=-1.}$

2 pts per ep.

$f(x) = x\ln(x+2) - \ln(x+2)$

$f'(x) = \ln(x+2) + \frac{x}{x+2} - \frac{1}{x+2} = \ln(x+2) + \frac{x-1}{x+2}$

1 pt for this work

$f'(1) = \ln(3) + 0 > 0$

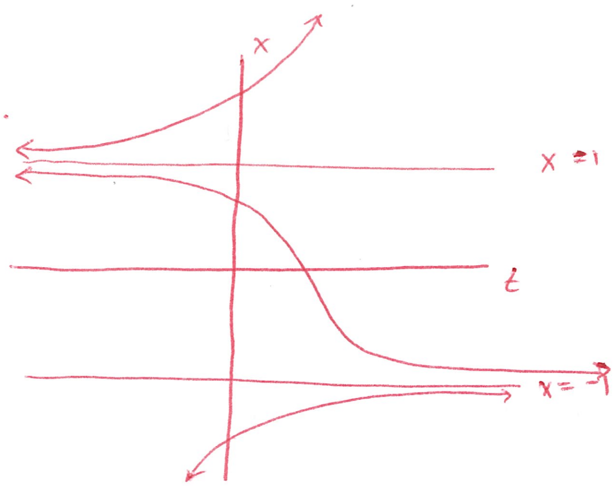
$f'(-1) = 0 - ? < 0.$

$\boxed{x=1 \text{ is unstable}}$

$\boxed{x=-1 \text{ is stable}}$

2 pts per classification.

b.



2 pts per curve
- 1 for drawing a curve in region
- 1 for correct concavity.

(15 points) *Problem 8.* Given a linear system $\mathbf{Ax} = \mathbf{b}$ as follows

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 4 \\ 2x_1 + 5x_2 - x_3 = 5 \\ 3x_1 + 6x_2 - 3x_3 + 3x_4 = 12 \end{cases}$$

(i) Apply elementary row operations to simplify the augmented matrix to the **reduced row echelon form**.

(ii) Write the solutions of the linear system in a parametric form.

(i) $$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 4 \\ 2 & 5 & -1 & 0 & 5 \\ 3 & 6 & -3 & 3 & 12 \end{array}\right) \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 4 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_2+R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 5 & 10 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(+3) Row operations (+3) RREF (+2) Correct RREF

(ii) $x_1 = 10 + 3x_3 - 5x_4$, $x_2 = -3 - x_3 + 2x_4$,

$x_3 = s$, $x_4 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix} \qquad s, t \in \mathbb{R}$$

(+2) Equ'ns for $x_1, \dots, x_4$ (+1) Correct equ'ns

(+1) Parameters = free variables (+1) Correct parametric representation

(+2) Parametric representation