ColoState Fall 2016 Math 340 Final

Mon. 12/12/2016

Name: Answers CSUID: ____________

Section: All 9 sections

Problem	Score
1	10
2	10
3	10
4	10
5	10
6	10
7	10
8	10
9	10
10	10
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use **two** letter-size 2-sided cheat sheets for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The ODE $(e^y - \cos(xy))dx + (x - \cos(xy))dy = 0$ is an exact equation.

(ii) (T) (F) If $x(t)$ is a solution of the ODE $x''(t) + 4x'(t) + 3x(t) = 0$, then $\lim_{t \to +\infty} x(t) = 0$.

(iii) (T) (F) The ODE $x''(t) + 9x(t) = 0$ models a damped harmonic motion.

(iv) (T) (F) $x(t) = e^{-t}$ is a solution of the ODE $x'' - x' - 2x = 3e^{-t}$.

(v) (T) (F) The inverse Laplace transform of $\dfrac{1}{s^2+4}$ is $\sin(2t)$.

(10 points) *Problem 2.* Consider the autonomous ODE system

$$\begin{cases} x'(t) = f(x,y) = -2y + x(x^2 + 4y^2 - 4) \\ y'(t) = g(x,y) = \dfrac{x}{2} - y(x^2 + 4y^2 - 4) \end{cases}$$

Determine whether the following statements are True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) $x(t) = 2\cos(t), y(t) = \sin(t)$ is a solution of the ODE system.

(ii) (T) (F) The partial derivative $g_x(x,y)$ is continuous on the whole plane $\mathbb{R}^2$.

(iii) (T) (F) There exists a unique solution curve on the phase plane that passes through the point $A(0,1)$.

(iv) (T) (F) There exists a unique solution curve on the phase plane that passes through the point $B(1,0)$.

(v) (T) (F) There is a solution curve on the phase plane that passes through both points $A(0,1)$ and $B(1,0)$.

(10 points) *Problem 3.* Consider an autonomous ODE $x'(t) = (1+x)(3-x)$.

(i) Find all equilibrium points and classify each of the equilibrium points as asymptotically stable or asymptotically unstable. Show your work.

(ii) Sketch three representative solution curves in the tx-plane. (*Remark: They should not be straight lines.*)

(iii) If $x(t)$ is a solution of the ODE and $x(0) = 0$, find $\lim_{t\to+\infty} x(t)$.

(i) Eq. pts at $x' = 0$

$(1+x)(3-x) = 0$

$1+x = 0, \quad 3-x = 0$

$\boxed{\text{Eq. pts } x = -1, \; x = 3}$ 2 pts (1 each)

$f(x) = (1+x)(3-x)$

$= 3 + 2x - x^2$

$= -x^2 + 2x + 3$

$f'(x) = -2x + 2$

1st deriv. test:

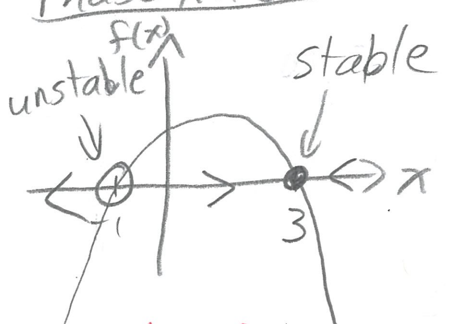
$f'(-1) = -2(-1) + 2 = 4 > 0$

$\boxed{\therefore x = -1 \text{ is unstable}}$

$f'(3) = -2(3) + 2 = -4 < 0$

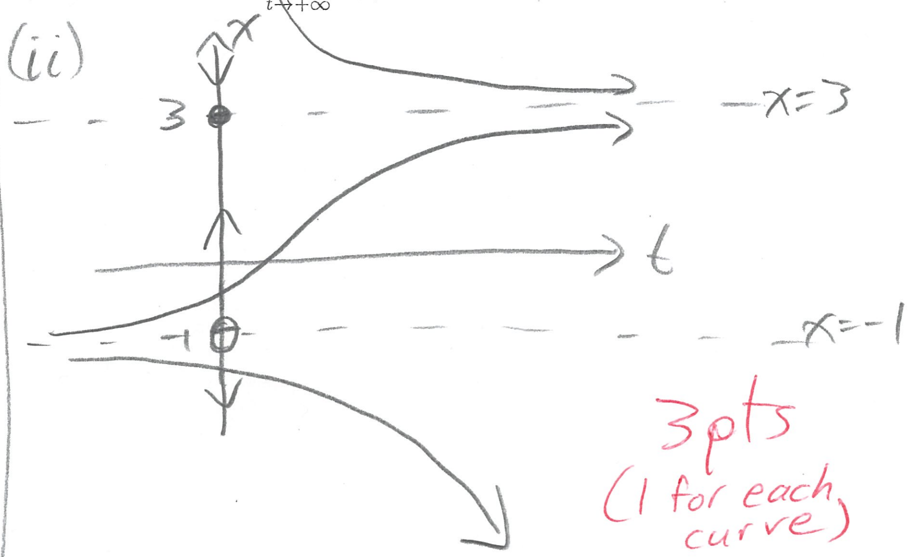
$\boxed{\therefore x = 3 \text{ is stable}}$

OR Phase line



3 pts $\left\{ \begin{array}{l} 2 \text{ for stability} \\ 1 \text{ for 1st deriv. test or phase line} \end{array} \right.$

(ii)



3 pts (1 for each curve)

(iii) If $x(t)$ starts at the origin, as $t \to \infty$ it will move towards the stable equilibrium solution $x(t) = 3$.

Thus, $\lim_{t\to\infty} x(t) = 3$.

2 pts

10.

(10 points) *Problem 4.* Given a linear ODE: $\cos(t)x'(t) - \sin(t)x(t) = e^{-t}$,

(i) Find the general solution.

(ii) Find the specific solution satisfying $x(0) = 0$.

1). $\cos(t)\, x'(t) - \sin(t)\, x(t) = e^{-t}$

$$x'(t) - \tan(t)\, x(t) = \frac{e^{-t}}{\cos(t)} \quad +1 \text{ point}.$$

Integrating factor.

$$u = e^{-\int \tan(t)\, dt}.$$

$$u = e^{\ln|\cos(t)|} = \cos(t). \ (+2 \text{ point})$$

$$\int \frac{\sin(t)}{\cos(t)}\, dt = \int \frac{-1}{w}\, dw = -\ln|w| = -\ln|\cos(t)|.$$

Total 6.

$$x(t) = \frac{1}{\cos(t)} \int \frac{\cancel{\cos(t)}\, e^{-t}}{\cancel{\cos(t)}}\, dt. \quad +1 \text{ pt}.$$

$$= \frac{1}{\cos(t)} \left[-e^{-t} + c \right] \quad + 2. \text{ pt}.$$

ii) $x(0) = \frac{1}{\cos(0)} \left[-1 + c \right] = 0.$

Total 4

$$-1 + c = 0.$$

$$c = 1. \quad + 3 \text{ pts}.$$

$$x(t) = \frac{1}{\cos(t)} \left[-e^{-t} + 1 \right] \quad + 1 \text{ pts}.$$

$\cos(t)dx) + (-\sin(t)\, x\, (-e^{-t}))\, dt = 0.$

$-\sin(t).\ f(-\sin(t)).$

(10 points) *Problem 5.* Find the solution for the initial value problem

$$y''(t) - 2y'(t) + y(t) = 0, \quad y(0) = -3, \quad y'(0) = 2.$$

Characteristic polynomial:

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1, 1.$

$\therefore$ General Solution: $y(t) = c_1 e^t + c_2 t e^t.$

$y'(t) = c_1 e^t + c_2 [t e^t + e^t]$

$\therefore \quad -3 = c_1$

$2 = c_1 + c_2 \Rightarrow c_2 = 2 - c_1 = 2 - (-3) = 5$

$\therefore \; y(t) = -3e^t + 5te^t$ 2

(10 points) *Problem 6.* Consider a 4th order constant coefficient linear ODE

$$y^{(4)}(t) - 2y''(t) + y(t) = 0.$$

(i) Find all roots of the characteristic equation;

(ii) Write down a general solution.

4 pts (i) $\lambda^4 - 2\lambda^2 + 1 = 0$

(1 pt for each root)

$(\lambda+1)^2(\lambda-1)^2 = 0$

$\lambda = -1$ mult 2

-1 if wrong multiplicities

$\lambda = 1$ mult 2

6 pts (ii)

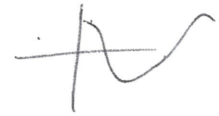
If $\lambda = \pm 1, \pm i$ then

$y(t) = C_1 e^t + C_2 t e^t + C_3 \sin(t) + C_4 \cos(t)$

1 pt for each correct term based on (i)

2 pts for correct linear combo

(+1) if didn't find roots but have correct Char. poly



(10 points) *Problem 7.* Consider an ODE system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix}$.

It is known that $\lambda = -1 - i$ is an eigenvalue of A and $\mathbf{w} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is an eigenvector assocaited with λ.

7 (i) Write down a **real-valued** general solution for the ODE system;

3 (ii) Classify the equilibrium at the origin (stable/unstable? center/nodal/saddle/spiral?)

← can be implicit.

i. $e^{(-1-i)t}\begin{pmatrix} i+1 \\ 1 \end{pmatrix} = e^{-t}\left(\cos(-t) + i\sin(-t)\right)\begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

+1

$= e^{-t}\begin{pmatrix} \cos(t) - i\sin(t) + i\cos(t) + \sin(t) \\ \cos(t) - i\sin(t) \end{pmatrix}$

$= e^{-t}\begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{pmatrix} + i e^{-t}\begin{pmatrix} \cos(t) - \sin(t) \\ -\sin(t) \end{pmatrix}$

$$\boxed{x(t) = \underset{+1}{C_1}\underset{+1}{e^{-t}}\underset{+1}{\begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{pmatrix}} + \underset{+1}{C_2}\underset{+1}{e^{-t}}\underset{+1}{\begin{pmatrix} \cos(t) - \sin(t) \\ -\sin(t) \end{pmatrix}}}$$

-OR-

$$x(t) = C_1 e^{-t}\begin{pmatrix} \cos(-t) - \sin(-t) \\ \cos(-t) \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} \cos(-t) + \sin(t) \\ \sin(-t) \end{pmatrix}$$

ii. Spiral sink, stable.

+2 +1

Key

(10 points) *Problem 8.* Consider an ODE system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} -1 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}$.

Determine whether the following statements are True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) 0 is an eigenvalue of A.

(ii) (T) (F) -1 is an eigenvalue of A.

(iii) (T) (F) $\mathbf{u}_1(t) = e^{-t}[-1, 0, 0]^T$ is a solution of the ODE system.

(iv) (T) (F) $\mathbf{u}_2(t) = e^{-t}[t, -1, 2]^T$ is a solution of the ODE system.

(v) (T) (F) $\mathbf{u}_3(t) = [2, 1, 1]^T$ is a solution of the ODE system.

(i), (ii) T T

$$(-1-\lambda)\left[(1-\lambda)(-2-\lambda)+2\right] = 0$$

$$(-1-\lambda)\left[-\cancel{2}-\lambda+2\lambda+\lambda^2+\cancel{2}\right] = 0$$

$$\lambda(-1-\lambda)(1+\lambda) = 0 \longrightarrow \lambda = 0, \underset{\text{mult } 2}{-1}.$$

(iii) $\vec{u}_1' = -e^{-t}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = e^{-t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\vec{u}_1$ T

(iv) $\vec{u}_2' = e^{-t}\begin{pmatrix} 1-t \\ 1 \\ -2 \end{pmatrix} = A\vec{u}_2$.

(v) $\vec{u}_3' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq A\vec{u}_3 = \begin{pmatrix} -1 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

See F16 Exam 2

(10 points) *Problem 9.* Consider a 2nd order ODE

$$y'' - y' - 2y = e^{-t}.$$

Use the method of undetermined coefficients to find a $\boxed{\text{particular}}$ solution of the ODE.

$$\left[\begin{array}{l} y_p = ae^{-t} \\ y_p' = -ae^{-t} \\ y_p'' = ae^{-t} \end{array}\right. \Rightarrow \left.\begin{array}{c} ae^{-t} + ae^{-t} - 2ae^{-t} = e^{-t} \\ 0 = e^{-t} \end{array}\right]$$

Does not work.

Try: 1 $y_p = ate^{-t}$

1 $y_p' = ae^{-t} - ate^{-t}$

1 $y_p'' = -ae^{-t} - ae^{-t} + ate^{-t}$

$= -2ae^{-t} + ate^{-t}$

$\Rightarrow$ 1 $y_p'' - y_p' - 2y_p = e^{-t}$

$\Leftrightarrow$ 1 $-2ae^{-t} + \cancel{ate^{-t}} - ae^{-t} + \cancel{ate^{-t}} - \cancel{2ate^{-t}} = e^{-t}$ 1

$\Leftrightarrow$ 1 $-3ae^{-t} = e^{-t}$

$\Leftrightarrow$ 1 $-3a = 1 \Leftrightarrow$ 1 $\boxed{a = -\frac{1}{3}}$

$\Rightarrow \boxed{y_p = -\frac{1}{3}te^{-t}}$ 1

$$\left[\begin{array}{l} p(\lambda) = \lambda^2 - \lambda - 2 \stackrel{!}{=} 0 \\ \Leftrightarrow (\lambda - 2)(\lambda + 1) = 0 \\ \Rightarrow e^{2t}, e^{-t} \text{ are solutions} \end{array}\right.$$

(10 points) *Problem 10.* Use Laplace transform and inverse Laplace transform to find the solution to the ODE IVP:

$$y''(t) + 4y = \cos(t), \; y(0) = 1, \; y'(0) = 0.$$

Hint:

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{s^2+1} + \frac{-1}{3}\frac{s}{s^2+4}.$$

$$\mathcal{L}(y'' + 4y)(s) = \mathcal{L}(\cos)(s)$$

$$\mathcal{L}(y'')(s) + \mathcal{L}(4y)(s) = \frac{s}{s^2+1^2}$$

-1 if swapped

$$s^2\mathcal{L}(y)(s) - sy(0) - y'(0) + 4\mathcal{L}(y)(s) = \frac{s}{s^2+1}$$

$$s^2\mathcal{L}(y)(s) - s\cdot 1 - 0 + 4\mathcal{L}(y)(s) = \frac{s}{s^2+1}$$

$$s^2\mathcal{L}(y)(s) - s + 4\mathcal{L}(y)(s) = \frac{s}{s^2+1}$$

$$(s^2+4)\mathcal{L}(y)(s) = \frac{s}{s^2+1} + s$$

-1 if off by sign

$$\mathcal{L}(y)(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$$

$$= \frac{1}{3}\frac{s}{s^2+1} + \frac{-1}{3}\frac{s}{s^2+4} + \frac{s}{s^2+4}$$

$$= \frac{1}{3}\frac{s}{s^2+1} + \frac{2}{3}\frac{s}{s^2+4}$$

$\boxed{6 \text{ points}}$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{3}\frac{s}{s^2+1} + \frac{2}{3}\frac{s}{s^2+4}\right)(t)$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+1^2}\right)(t) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right)(t)$$

$$= \frac{1}{3}\cos t + \frac{2}{3}\cos 2t$$

$\boxed{4 \text{ points}}$