ColoState   Fall 2016   Math 340   Exam 2

Thu. 11/10/2016

Name: Answers   CSUID: 100/100

Section: ~~All~~ 9 Sections

Problem	Score
1	10
2	10
3	10
4	10
5	10
6	10
7	10
8	10
9	10
10	10
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use **one** letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) If $x(t)$ is a solution of the ODE $x''(t) + 4x'(t) + 3x(t) = 0$, then $\lim_{t\to+\infty} x(t) = 0$.

(ii) (T) (F) The function $x(t) = e^t + e^{-2t}$ is a solution to the 3rd order ODE $x'''(t) + 3x''(t) - 4x(t) = 0$.

(iii) (T) (F) Suppose $y_1(t) = 3te^{-t}$ and $y_2(t) = (1+t^2)e^{-t}$ are both solutions of a 2nd order ODE $y''(t) + p(t)y'(t) + q(t)y(t) = 0$. Then $y_1(t), y_2(t)$ form a fundamental set of solutions.

(iv) (T) (F) A 2×2 real matrix has at least one real eigenvalue.

(v) (T) (F) The ODE $x''(t) + 9x(t) = 0$ models a damped harmonic motion.

(i) General solution $X(t) = C_1e^{-t} + C_2e^{-3t} \to 0$ as $t \to +\infty$ (chara. eqn. $\lambda^2 + 4\lambda + 3 = 0$)
(ii) Chara. eqn. $\lambda^3 + 3\lambda^2 - 4 = 0$, both $1, -2$ are chara. roots.
(iii) $3te^{-t}, (1+t^2)e^{-t}$ are linearly independant.
(iv) $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ is a counterexample, $3 \pm 4i$ eigenvalues. See this exam Prob. 8 also
(v) This is a simple harmonic motion, see Textbook P. 158
See Textbook P. 160 for damped harmonic motion

(10 points) *Problem 2.* Consider the autonomous ODE system

$$\begin{cases} x'(t) = f(x,y) = -y + x(x^2 + y^2 - 1) \\ y'(t) = g(x,y) = x - y(x^2 + y^2 - 1) \end{cases}$$

Determine whether the following statements are True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) $x(t) = \cos(t), y(t) = \sin(t)$ is a solution of the ODE system.

(ii) (T) (F) The partial derivative $f_x(x,y)$ is continuous on the whole plane $\mathbb{R}^2$.

(iii) (T) (F) There exists a unique solution curve on the phase plane that passes through the point $A(1,0)$.

(iv) (T) (F) There exists a unique solution curve on the phase plane that passes through the point $B(-0.5, 0)$.

(v) (T) (F) There is a solution curve on the phase plane that passes through both points $A(1,0)$ and $B(-0.5,0)$.

2 pts each

② (i) $x(t) = \cos(t)$, $y(t) = \sin(t)$

$\boxed{\text{(i) TRUE}}$

$$f(x(t), y(t)) = -\sin(t) + \cos(t)\underbrace{(\cos^2(t) + \sin^2(t)}_{1} - 1)$$
$$= -\sin(t)$$
$$= x'(t) \checkmark$$

$$g(x(t), y(t)) = \cos(t) - \sin(t)\underbrace{(\cos^2(t) + \sin^2(t)}_{1} - 1)$$
$$= \cos(t)$$
$$= y'(t) \checkmark$$

(ii) $f_x(x,y) = \frac{\partial}{\partial x}(x^3 + xy^2 - x) = 3x^2 + y^2 - 1$

polynomial, so continuous on all of $\mathbb{R}^2$

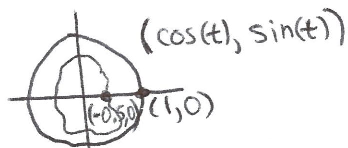
$\boxed{\text{(ii) TRUE}}$

(iii) $f_y = -1 + 2xy$, $g_x = 1 - 2xy$, $g_y = -x^2 - 3y^2 + 1$,

f_x, f, g all continuous on all of $\mathbb{R}^2$

$\boxed{\text{(iii) TRUE}}$

$\boxed{\text{(iv) TRUE}}$

(v) 

(cos(t), sin(t)) is the unique solution curve that passes through (1,0). The point (-0.5,0) lies inside the unit circle, so the unique solution curve passing through (-0.5,0) cannot pass through (1,0) or else the solution curves cross.

$\boxed{\text{(V) FALSE}}$

(10 points) *Problem 3.* In an experiment,

- A 5-kg mass is attached to a spring;
- The displacement of the mass-spring equilibrium from the spring equilibrium is measured to be $0.75m$;
- The mass is then displaced $0.36m$ upward from the mass-spring equilibrium;
- Then the system is given a sharp downward tap, imparting an instantaneous downward velocity of $0.45m/s$.

Assume there is no damping present. Set up (but **do not solve**) an initial value problem for the resulting motion.

$$K = \frac{mg}{x_0} = \frac{196}{3} \simeq 65.3$$

[+5 pts]

$$\boxed{\begin{aligned} x'' + \frac{K}{m}x - g &= 0. \\ x(0) &= 0.39 \\ x'(0) &= 0.45 \end{aligned}}$$

+5 pts.

-2 if wrong sign with initial conditions

-2 if miss g.

$$\boxed{\begin{aligned} y'' + \frac{K}{m}y &= 0. \\ y(0) &= -0.36 \\ y'(0) &= 0.45. \end{aligned}}$$

-2 if wrong sign or wrong initial conditions

(−1) for arithmetic mistake

(10 points) *Problem 4.* Find a **real-valued** general solution for the ODE

$$y''(t) + 6y'(t) + 10y(t) = 0.$$

Method 1 $\Rightarrow \lambda^2 + 6\lambda + 10 = 0 \Rightarrow \left(\lambda = \frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm \frac{\sqrt{-4}}{2} = -3 \pm i\right)$

$z(t) = e^{(-3\pm i)t} = e^{-3t}\cdot e^{\pm it} = e^{-3t}(\cos(t) \pm i\sin(t)) = e^{-3t}\cos(t) \pm i e^{-3t}\sin(t)$

$\Rightarrow$ real solutions $\boxed{y_1(t) = e^{-3t}\cos(t)}$, $\boxed{y_2(t) = e^{-3t}\sin(t)}$

$\Rightarrow \boxed{y(t) = C_1 e^{-3t}\cos(t) + C_2 e^{-3t}\sin t}$

(−2 if not c_1, c_2)
(−3 if not real sols.)
(−2 if vector solution)

+2 characteristic equation
+3 eigenvalues λ
+3 expansion of e^{λ}
+2 pick real/imag parts ⇒ general $y(t)$ / y_1, y_2 or

Method 2 $\begin{cases} x_1 = y_1 \\ x_2 = y' \end{cases} \Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -10x_1 - 6x_2 \end{cases} \Rightarrow \vec{x}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -10 & -6 \end{pmatrix}}_{A} \vec{x}$

$\Rightarrow P_A(\lambda) = \cancel{(-1)^2}\begin{vmatrix} -\lambda & 1 \\ -10 & -6-\lambda \end{vmatrix} = (-\lambda)(-6-\lambda) - (-10) = 6\lambda + \lambda^2 + 10 \Rightarrow \lambda = -3 \pm i$ eigenvalues

null $\begin{pmatrix} 3-i & 1 \\ -10 & \underbrace{-6+3}_{-3}-i \end{pmatrix} \Rightarrow \begin{cases} (3-i)x_1 + x_2 = 0 \\ -10x_1 + (-3-i)x_2 = 0 \end{cases}$, if $x_1 = -1 \Rightarrow x_2 = 3-i$ $\Rightarrow +10 + (-3-i)(3-i) =$

$= 10 + (-i+3)(-i-3) = 10 + (-i)^2 - 3^2 =$
$= 10 + (-1) - 9 = 0$
so $x_1 = -1$, $x_2 = 3-i$, valid solution
or $x_1 = 1$, $x_2 = -3+i$ 〃 〃

$\Leftarrow$

$\Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -3+i \end{pmatrix}$ eigenvector

$\therefore \vec{z}(t) = e^{(-3+i)t}\begin{pmatrix} 1 \\ -3+i \end{pmatrix} =$

$= e^{-3t}(\cos(t) + i\sin(t))\underbrace{\begin{pmatrix} 1 \\ -3+i \end{pmatrix}}_{\left[\begin{pmatrix} 1 \\ -3 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]} = e^{-3t}\left[\cos t\begin{pmatrix} 1 \\ -3 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\sin(t)\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right] =$

$= e^{-3t}\left[\begin{pmatrix} \cos(t) \\ -3\cos(t) - \sin(t) \end{pmatrix} + i\begin{pmatrix} \sin(t) \\ \cos(t) - 3\sin(t) \end{pmatrix}\right]$

picking $\vec{y}_1 = e^{-3t}\begin{pmatrix} \cos(t) \\ -3\cos(t) - \sin(t) \end{pmatrix}$, $\vec{y}_2 = e^{-3t}\begin{pmatrix} \sin(t) \\ \cos(t) - 3\sin(t) \end{pmatrix}$ we get $x_1 = y$, the 1st component as the real solutions for $y(t)$, like in method 1.

$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 \Rightarrow \boxed{y(t) = C_1 e^{-3t}\cos(t) + C_2 e^{-3t}\sin(t)}$

+2 eigenvalue
+3 eigenvector
+3 expansion $e^{\lambda t}\vec{v}_1$
+2 pick real/imag.

(10 points) *Problem 5.* Find the solution for the initial value problem

$$y''(t) - 2y'(t) + y(t) = 0, \quad y(0) = 2, \quad y'(0) = -3.$$

Charateristic polynomial:

1 $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

2 Roots: $\lambda = 1, 1$.

$\therefore \; y_1(t) = e^t \;;\; y_2(t) = te^t.$

2 $\therefore \; y(t) = c_1 e^t + c_2 te^t.$

$y'(t) = c_1 e^t + c_2 [te^t + e^t]$

2 $y(0) = 2 = c_1$

$y'(0) = -3 = c_1 + c_2$

2 $\Rightarrow c_2 = -3 - 2 = -5$.

1 $y(t) = 2e^t - 5te^t$.

(10 points) *Problem 6.* Consider a 4th order constant coefficient linear ODE

$$y^{(4)}(t) - y(t) = 0.$$

(i) Find all roots of the characteristic equation; 5pts

(ii) Write down a **real-valued** general solution. 5pts

(i) Characteristic polynomial: $p(\lambda) = \lambda^4 - 1$ 1pt.

$p(\lambda) = 0 \Leftrightarrow \lambda^4 - 1 = 0 \Leftrightarrow (\lambda^2-1)(\lambda^2+1) = 0$

$\Leftrightarrow \lambda = \pm 1$ or $\lambda \pm i$ 4pt.

(ii) Fundamental set of (real) solutions:

$y_1(t) = e^t$ $\quad$ $y_3(t) = \cos t$

$y_2(t) = e^{-t}$ $\quad$ $y_4(t) = \sin t$

4pt. (1 pt each)

~~$y_4($~~

General solution:

$y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t.$ 1/pt

(10 points) *Problem 7.* Consider a second order ODE

$$x''(t) + 2x'(t) = 3x - x^3.$$

(i) Convert the above ODE into a first-order ODE system;

(ii) Find all equilibria of the first-order ODE system.

(i) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} u_2 \\ 3u_1 - u_1^3 - 2u_2 \end{pmatrix}$ 4 Points

(ii) $\left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} u_2 \\ 3u_1 - u_1^3 - 2u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| u_2 = 0, 3u_1 - u_1^3 = 0 \right\}$

$= \left\{ \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \right\}$

2 Points 2 Points 2 Points

@

(10 points) *Problem 8.* Consider an ODE system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$.

It is known that $\lambda = 5 + 2i$ is an eigenvalue of A and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix}$ is an eigenvector assocaited with λ.

7 (i) Write down a **real-valued** general solution for the ODE system;

3 (ii) Classify the equilibrium at the origin (stable/unstable? center/nodal/saddle/spiral?)

• 1/3 if no 2

$$e^{\lambda t} = e^{5t}(\cos(2t) + i\sin(2t)) \quad +3$$

$$e^{\lambda t}\vec{w} = e^{5t}\begin{pmatrix} \cos(2t) + i\sin(2t) \\ \cos(2t) + i\sin(2t) - 2i\cos(2t) + 2\sin(2t) \end{pmatrix}$$

5/7: If two particular solutions.

$$= e^{5t}\begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + i e^{5t}\begin{pmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}$$

$$\boxed{\vec{x}(t) = \underbrace{C_1 e^{5t}}_{+1}\underbrace{\begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix}}_{+1} + \underbrace{C_2 e^{5t}}_{+1}\underbrace{\begin{pmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}}_{+1}}$$

Spiral source

+2

unstable

+1

• If they include i in ~~an~~ answer −1.

$$\vec{x}(t) = C_1 e^{5t}\begin{pmatrix} \cos(-2t) \\ \cos(-2t) + 2\sin(-2t) \end{pmatrix} + C_2 e^{5t}\begin{pmatrix} \sin(-2t) \\ \sin(-2t) + 2\cos(-2t) \end{pmatrix}$$

$$\begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} - C_2 e^{5t}\begin{pmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}$$

Exam 2

(10 points) *Problem 9.* Given an order 3 matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. It is known that 1, 4 are two eigenvalues of A. Find one eigenvector for each of the two eigenvalues.

$\lambda = 1$

5

$(A - \lambda I)\vec{v} = \vec{0}$ $\quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

2

$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow$ $v_3 = t, \quad v_2 = s.$

$v_1 + s + t = 0 \Rightarrow v_1 = -s - t$

3

$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, e-vector $= \boxed{s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cancel{+} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$\lambda = 4$

5

$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow[R_3 - R_2]{R_3 \to} \left(\begin{array}{cccc} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$

$\xrightarrow{R_2 \to 2R_2 + R_1} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \xrightarrow[R_3 + R_2]{R_3 \to} \left(\begin{array}{cccc} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

3

(-1) for row-reduce mistake

$w_3 = t, \quad -3w_2 + 3t = 0$

$w_2 = t$

$-2w_1 + t + t = 0 \Rightarrow -2w_1 = -2t$

$\Rightarrow w_1 = t$

$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \underbrace{\boxed{t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}}_{\text{evector}}$ 2

-2 if wrong e-vec

(10 points) *Problem 10.* Consider an ODE system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} -1 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}$.

It is known that

- 0 is an eigenvalue with algebraic multiplicity 1 and has an eigenvector $[-2, -1, 1]^T$;
- -1 is an eigenvalue with algebraic multiplicity 2 and has an eigenvector $[1, 0, 0]^T$.

Find a general solution for the ODE system.

Two solutions from given eigenpairs $\quad +2pt$

$$\vec{x}_1(t) = e^{0t}\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{x}_2(t) = e^{-t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Third solution requires a generalized eigenvector for $\lambda_2 = -1$

Approach 1: Solve $(A+I)^2\vec{v} = \vec{0}$ $\quad +2pt$

$$(A+I)^2 = \begin{bmatrix} 0 & 5 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 5 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

$R_1 - 2R_2 \to R_2$

$R_1 + 2R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

v_1, v_3 free

$4v_2 + 2v_3 = 0$

$\Rightarrow v_2 = -\frac{1}{2}v_3$

$$\Rightarrow \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ -\frac{1}{2}v_3 \\ v_3 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

choose $v_1 = 0$, $v_3 = 2$ $\quad +3pt$

$\Rightarrow \vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ (or other choices of v_1, v_3 to be so that $\vec{v}$ is lin. ind. with given eigvecs)

$$\Rightarrow \vec{x}_3(t) = e^{-t}\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + t(A+I)\vec{v}\right)$$

$$= e^{-t}\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 & 5 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right)$$

$+2pt$

$$= e^{-t}\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= e^{-t}\begin{bmatrix} t \\ -1 \\ 2 \end{bmatrix}$$

General solution $\quad +1pt$

$$\boxed{\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)}$$

with $\vec{x}_1, \vec{x}_2, \vec{x}_3$ defined above

Check lin. ind.

$$W(0) = \begin{vmatrix} -2 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= -1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = -1(-2+1) = 1 \neq 0$$

so $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are lin. ind. and form a fund. set of solns