ColoState   Fall 2016   Math 340   Exam 1

Thu. 10/06/2016

Name: Answers   CSUID: 100/100

Section: #1~9.

Problem	Score
1	10
2	15
3	15
4	15
5	15
6	15
7	15
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use **one** letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The constant function $y(x) \equiv \pi/2$ is a solution to $y'(x) = 2x\cos^2(y)$.

(ii) (T) (F) The initial value problem $tx'(t) = x + 3t^2, x(0) = 1$ does not have a solution.

(iii) (T) (F) The ODE $(x + 3x^3\sin(y))dx + x^4\cos(y)dy = 0$ is exact.

(iv) (T) (F) For a given logistic population model $\dfrac{dP(t)}{dt} = \left(1 - \dfrac{P}{K}\right)P$, the rate at which the population is increasing is at its greatest when the population is at one-third of its carrying capacity K.

(v) (T) (F) Let $\mathbf{u} = (2,2,1), \mathbf{v} = (1,1,2), \mathbf{w} = (4,4,2)$ be three vectors. Then the subspace spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ has dimension 3.

(15 points) *Problem 2.* A ball is thrown upwards into the air at an initial height $y_0 = 20$ ft with an initial velocity $v_0 = 8$ ft/s. Ignore the air resistance.

(i) Find the maximal height reached by the ball.

(ii) At what time does this event (*reaching its maximal height*) occur?

(iii) When does the ball hit the ground? Answer to this part should be simplified.

Hint: $g = 32$ *ft*/*s*^{2}.

$$\frac{dv}{dt} = -g.$$

3

$$\therefore v(t) = -32t + 8 \;/\; v(t) = -gt + v_0 \;/\; v(t) = at + v_0$$

$$y(t) = -16t^2 + 8t + 20 \;/\; y(t) = \frac{-gt^2}{2} + v_0 t + y_0.$$

(i) when ball reaches max height $v = 0$.

$$\therefore -32t + 8 = 0 \Rightarrow \boxed{t = \frac{1}{4}\,s}$$

4

$$\therefore y\left(\frac{1}{4}\right) = -16\left(\frac{1}{4}\right)^2 + 8\left(\frac{1}{4}\right) + 20 = \boxed{21 \text{ ft}}$$

4

(ii) $t = \frac{1}{4}\,s$ from previous work.

(iii) when ball hits ground, $y = 0$.

$$\therefore -16t^2 + 8t + 20 = 0.$$

4

$$t = \frac{2 \pm \sqrt{84}}{8}$$

$$\boxed{t = \frac{1 \pm \sqrt{21}}{4}.}$$

(15 points) *Problem 3.* Consider the 1st order ODE:

$$x'(t) - \tan(t)x(t) = \sin(t).$$

(i) Find its general solution.

(ii) Find the particular solution satisfying $x\left(\frac{\pi}{4}\right) = \frac{3}{4}\sqrt{2}$.

$$x'(t) = \underbrace{\tan(t)}_{a(t)} x(t) + \underbrace{\sin(t)}_{f(t)}$$

$$X_h(t) = e^{\int a(t)dt} = e^{\int \tan(t)dt}$$

$$= e^{-\int \frac{d\cos(t)}{\cos(t)}}$$

$$= e^{-\ln|\cos(t)|}$$

$$= \boxed{\sec(t)}$$ 5 points

(i) $$x(t) = X_h(t) \int \frac{f(t)}{X_h(t)} dt$$

$$= \sec(t) \int \frac{\sin(t)}{\sec(t)} dt$$

$$= \sec(t) \int \sin(t) d\sin(t)$$

$$= \boxed{\sec(t)\left(\frac{\sin^2(t)}{2} + c\right)}$$ 5 points

(ii) $$x\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right)\left(\frac{\sin^2\left(\frac{\pi}{4}\right)}{2} + c\right)$$

$$= \sqrt{2}\left(\frac{1}{4} + c\right) = \frac{3}{4}\sqrt{2}$$

$$\Longrightarrow \frac{1}{4} + c = \frac{3}{4}$$

$$\Longrightarrow \boxed{c = \frac{1}{2}}$$

Particular solution
$$X(t) = \frac{\sin^2(t) + 1}{2\cos(t)}$$

5 points

(15 points) *Problem 4.* It is known that the ODE

$$Pdx + Qdy = (xy - 2)dx + (x^2 - xy)dy = 0$$

is not exact but has an integrating factor that depends only on x.

5 pts (i) Find such an integrating factor.

10 pts (ii) Find the general solution of the given ODE.

(i) $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ 2 pts

$= \frac{1}{x^2 - xy}\left(x - (2x - y)\right)$

$= -\frac{1}{x}$

$\mu(x) = e^{\int h(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = \boxed{\frac{1}{x}}$ 3 pts

(ii) ODE. $\frac{xy-2}{x}dx + \frac{(x^2 - xy)}{x}dy = 0$ 1 pt

$\left(y - \frac{2}{x}\right)dx + (x-y)dy = 0$

Check $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(y - \frac{2}{x}\right) = 1 = \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x-y)$

Exact ✓

$\frac{\partial F}{\partial x} = y - \frac{2}{x}, \quad \frac{\partial F}{\partial y} = x - y$

$F(x,y) = yx - 2\ln|x| + \phi(y)$

$\frac{\partial F}{\partial y} = x + \phi'(y) = x - y$ 8 pts

$\phi'(y) = -y$

$\phi(y) = -\frac{1}{2}y^2$

$\boxed{F(x,y) = yx - 2\ln|x| - \frac{1}{2}y^2 = C}$ 1 pt

(15 points) *Problem 5.* There are two parts in this problem.

(i) Show that $x_0(t) \equiv 0$ and $x_1(t) = \dfrac{t^4}{16}$ are both solutions to the initial value problem $x'(t) = t\sqrt{x}, x(0) = 0$.

(ii) Explain why this fact does not contradict the Uniqueness Theorem.

6pts

~~6pts~~ (i) For $x_0 \equiv 0$: $\frac{d}{dt}(x) = \frac{d}{dt}(0) = 0 = t\sqrt{0} = t\sqrt{x}$.

For $x_1 = \frac{t^4}{16}$: $\frac{d}{dt}(x) = \frac{d}{dt}\left(\frac{t^4}{16}\right) = \frac{t^3}{4} = t\sqrt{\frac{t^4}{16}} = t\sqrt{x}$.

- 3pts for each verification.
- Partial credit if they only took derivative or plugged x_i into $t\sqrt{x}$.

9pts

~~9pts~~ (ii) $f(t,x) = t\sqrt{x}$ and $\frac{\partial f}{\partial x} = \frac{t}{2\sqrt{x}}$.

$\frac{\partial f}{\partial x}$ is not continuous, let alone defined, at $x = 0$. Therefore, $\frac{\partial f}{\partial x}$ is not continuous at $(t, 0)$. The Uniqueness Theorem does not apply.

- 4pts (2pts each) for identifying $f(t,x)$ and $\frac{\partial f}{\partial x}$.
- 5pts for the explanation. Partial credit given.

(15 points) *Problem 6.* Consider an autonomous ODE $x'(t) = 1 - x^4$.

3 (i) Find all equilibrium points.

6 (ii) Classify each of the equilibrium points as asymptotically stable or asymptotically unstable. Show your work.

6 (iii) Sketch three representative solution curves in the tx-plane.
(*Remark: They should not be straight lines.*)

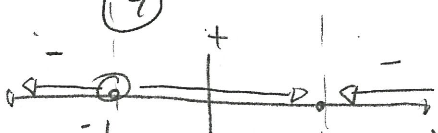
+1

(i) $x' = 1 - x^4 \iff (1-x^2)(1+x^2) = 0 \iff (1-x)(1+x)(x^2+1) = 0$

$\iff x = \pm 1$ $\boxed{\begin{matrix} x=-1 \\ x=1 \end{matrix}}$

2 one point for each

(ii) 4



or $f(x) = 1 - x^4$

4 $f'(x) = -4x^3$

points for analysis either way

$f'(-1) = -4(-1)^3 = 4 > 0 \rightarrow x = -1$ unstable.

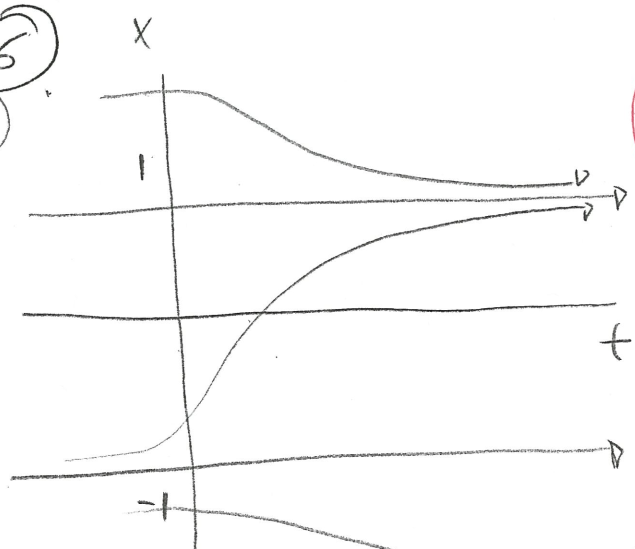
$f'(1) = -4(1)^3 = -4 < 0 \rightarrow x = 1$ is asy. stable

2 points for answer

$x = -1$ is unstable

$x = 1$ is asymptotically ~~unstable~~ stable.

(iii) 6



2/3 for each curve

(15 points) *Problem 7.* Given a linear system

$$\begin{cases} x_1 + 2x_2 + 2x_3 - 2x_4 = 3 \\ x_1 + 2x_2 - 2x_3 + 4x_4 = 2 \\ -x_1 - 3x_2 = 2 \end{cases}$$

6pts arithmetic
1 REF

REF → 7pts

→ +2pts

9pts (i) Apply row operations to simplify the augmented matrix to the reduced row echelon form (RREF).

6pts (ii) Write the solutions of the linear system in the parametric form.

3pts a soln as vector
3pts parametric

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -2 & 3 \\ 1 & 2 & -2 & 4 & 2 \\ -1 & -3 & 0 & 0 & 2 \end{array}\right]$$

$R_3 + R_1 \to R_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -2 & 3 \\ 0 & 0 & -4 & 6 & -1 \\ 0 & -1 & 2 & -2 & 5 \end{array}\right]$$

$R_2 \Leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -2 & 3 \\ 0 & -1 & 2 & -2 & 5 \\ 0 & 0 & 4 & 6 & -1 \end{array}\right]$$

$-R_2 \to R_2, \; -\frac{1}{4}R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -2 & 3 \\ 0 & 1 & -2 & 2 & -5 \\ 0 & 0 & 1 & -3/2 & 1/4 \end{array}\right]$$

$R_2 + 2R_3 \to R_2$
$R_1 - 2R_3 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 5/2 \\ 0 & 1 & 0 & -1 & -9/2 \\ 0 & 0 & 1 & -3/2 & \frac{1}{4} \end{array}\right]$$

$R_1 - 2R_2 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 11.5 \\ 0 & 1 & 0 & -1 & -9/2 \\ 0 & 0 & 1 & -3/2 & 1/4 \end{array}\right]$$

RREF

(ii) $\vec{x} = \begin{bmatrix} 11.5 \\ -4.5 \\ 0.25 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -3/2 \\ -1 \end{bmatrix}$