ColoState Fall 2013 Math 340 Final

Mon. 12/16/2013

Name: Answers CSUID: ____________

Section: ________

Problem	Score
1	20
2	10
3	15
4	10
5	15
6	15
7	15
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, notes, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use two double-sided Cheat Sheets for this exam.

Good luck!

(20 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) $\boxed{\text{(T)}}$ (F) The function $x(t) = \tan(t)$ is a solution to the ODE $x'(t) = 1 + x(t)^2$.

(ii) (T) $\boxed{\text{(F)}}$ The equation $(xy - 1)dx + (x^2y - xy^2)dy = 0$ is exact.

(iii) $\boxed{\text{(T)}}$ (F) Suppose $\lambda_i(i = 1, 2)$ are distinct eigenvalues of a matrix $\mathbf{A}$ and $\mathbf{v}_i(i = 1, 2)$ are eigenvectors corresponding respectively to λ_i. Then $e^{\lambda_i t}\mathbf{v}_i(i = 1, 2)$ are two linearly independent solutions of the ODE system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

(iv) (T) $\boxed{\text{(F)}}$ For the given matrix $\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$, the geometric multiplicity of the eigenvalue $\lambda = -2$ is three.

(v) $\boxed{\text{(T)}}$ (F) For the given matrix $\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$, it has two different complex eigenvalues.

(vi) (T) $\boxed{\text{(F)}}$ Consider the ODE system $\begin{cases} x'(t) = -2x - y \\ y'(t) = x - 2y \end{cases}$. The equilibrium point at the origin is a center.

(vii) (T) $\boxed{\text{(F)}}$ The ODE $x''(t) + 10x'(t) + 16x(t) = 0$ describes an underdamped harmonic motion.

$2c = 10$ $\quad w_0 = 4 \Rightarrow$ overdamped

$c = 5$

(viii) (T) $\boxed{\text{(F)}}$ The curve $(x(t), y(t)) = (\cos(t), 2\sin(t))$ is a solution curve of the autonomous ODE system $\begin{cases} x'(t) = -2y + x(x^2 + 4y^2 - 4) \\ y'(t) = \frac{1}{2}x - y(x^2 + 4y^2 - 4) \end{cases}$

(ix) (T) $\boxed{\text{(F)}}$ Since the Laplace transform of $f(t)$ is $\mathcal{F}(s)$, the Laplace transform of $f'(t)$ is $s\mathcal{F}(s)$.

(x) (T) $\boxed{\text{(F)}}$ The inverse Laplace transform of $\frac{s}{s^2+1}$ is $\sin(t)$. Its $\cos(t)$

(viii) $x'(t) = -\sin(t)$ $\quad y'(t) = 2\cos(t)$

$-4\sin t + \cos t(\cos^2 t + 4(4\sin^2 t) - 4)$

(i) $x'(t) = \sec^2(t)$

$1 + \tan^2(t) = \sec^2(t)$ ✓

(ii) $M_y = x \quad N_x = 2xy - y^2$

$\neq$

(iv) $\det(A - \lambda I) = \det\begin{bmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & -1 \\ 0 & 1 & -2-\lambda \end{bmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix}$

$= (-2-\lambda)[(-2-\lambda)(-2-\lambda) + 1]$

$= (-2-\lambda)[4 + 4\lambda + \lambda^2 + 1]$

$= (-2-\lambda)(\lambda^2 + 4\lambda + 5)$

$= 0$

$\lambda_1 = -2 \quad \lambda_2 = \frac{-4 \pm \sqrt{16 - 4(5)}}{2} = -2 \pm i$

(vi) $A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$

$\det(A - \lambda I) = \det\begin{bmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix} = (-2-\lambda)^2 + 1$

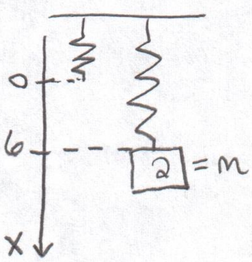
$= \lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = -2 \pm i$

(10 points) *Problem 2.* Consider a mass-spring system: Key

- The mass weighs 2 kg;
- The mass stretches the spring by 0.06 m before the motion starts;
- A damping force numerically equal two times the instantaneous velocity acts on the system;
- The gravity acts on the system also;
- The mass is given an initial tap at the mass-spring equilibrium with an upward velocity 0.1 m/s.

Set up but do not solve an ODE and two initial conditions to describe the motion.

+2 $mx'' + cx' + kx = mg$



+1 $m = 2$

+1 $c = 2 \leftarrow (.2 \text{ ok})$

+2 $k = \dfrac{mg}{x_0} = \dfrac{2(9.8)}{.06}$

+1
$$\boxed{\begin{gathered} 2x'' + \underbrace{2x'}_{.2 \text{ ok}} + \left(\frac{2(9.8)}{.06}\right)x = 2(9.8) \\ x(0) = .06 \quad (+1) \\ x'(0) = -.1 \quad (+2) \end{gathered}}$$

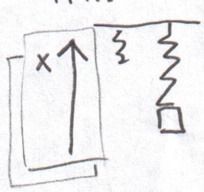
• -1 if unclear constants

• -1 if wrong simplification of constants

or

$$\boxed{\begin{gathered} 2y'' + 2y' + \left(\frac{2(9.8)}{.06}\right)y = 0 \\ y(0) = 0 \\ y'(0) = -.1 \end{gathered}}$$

• if...



then

$$\boxed{\begin{gathered} 2x'' + 2x' + \left(\frac{2(9.8)}{.06}\right)x = -2(9.8) \\ x(0) = -.06 \\ x'(0) = .1 \end{gathered}}$$

or

$$\boxed{\begin{gathered} 2y'' + 2y' + \left(\frac{2(9.8)}{.06}\right)y = 0 \\ y(0) = 0 \\ y'(0) = .1 \end{gathered}}$$

(15 points) *Problem 3.* It is known that the ODE $xydx + (2x^2 + 3y^2 - 20)dy = 0$ is not exact but has an integrating factor that depends only on y.

8pts (i) Find the integrating factor.

7pts (ii) Find the general solution to the ODE.

$P = xy, \quad Q = 2x^2 + 3y^2 - 20$

(i) $\mu(y) = e^{-\int g(y)dy}$ for $g(y) = \frac{1}{P}(P_y - Q_x) = \underbrace{\frac{1}{xy}(x - 4x) = -\frac{3}{y}}_{3\text{pts}}$

$\underbrace{\mu = e^{-\int -\frac{3}{y}dy}}_{3\text{pts}} = e^{3\ln(y)} = \underbrace{y^3}_{2\text{pts}}$

check: $xy^4dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$

$P_y = 4xy^3, \quad Q_x = 4xy^3 \quad P_y = Q_x$ ✓

(ii) Way 1

$F(x,y) = \overbrace{\int P dx = \int xy^4 dx}^{1\text{pt}} = \overbrace{\frac{1}{2}x^2y^4 + \phi(y)}^{1\text{pt}}$

$\frac{\partial F}{\partial y} = \underbrace{2x^2y^3 + \phi'(y)}_{1\text{pt}} = \underbrace{2x^2y^3 + 3y^5 - 20y^3}_{1\text{pt}}$

$\underbrace{\phi'(y) = 3y^5 - 20y^3}_{1\text{pt}} \Rightarrow \underbrace{\phi(y) = \frac{1}{2}y^6 - 5y^4}_{1\text{pt}}$

Way 2

$F(x,y) = \int Q dy = \int (2x^2y^3 + 3y^5 - 20y^3)dy$

$= \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + \phi(x)$

$\frac{\partial F}{\partial x} = xy^4 + \phi'(x) = xy^4 \Rightarrow \phi'(x) = 0$

$F(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$ ← 1pt

* Max 3 pts for (ii) with incorrect (i)

(10 points) *Problem 4.* True or False, circle your answer (2 points for each item, no partial credit). Given a function $f(t,x) = \sqrt[3]{t}\sqrt{x}$ and consider the ODE $x'(t) = f(t,x)$.

(i) (T) (F)
$f(t,x)$ and $\frac{\partial f}{\partial x}$ are both continuous on the whole (t,x) plane.

(ii) (T) (F)
$f(t,x)$ and $\frac{\partial f}{\partial x}$ are both continuous in the region $R = \{(t,x) : t > 0, x > 0\}$.

(iii) (T) (F)
The initial value problem $x' = f(t,x), x(0) = 1$ has a unique solution.

(iv) (T) (F)
The initial value problem $x' = f(t,x), x(1) = 0$ has a unique solution.

(v) (T) (F)
The initial value problem $x' = f(t,x), x(1) = 1$ has a unique solution.

(15 points) *Problem 5.* Given an ODE system $\mathbf{x}'(t) = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$, find the general solution.

Upper triangular system, eig. values on diagonal. $\lambda_1 = \lambda_2 = \lambda_3 = 2$ +2 pts

or det $(A - \lambda I) = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 2. \Rightarrow$ A.M = Algebraic multiplicity = 2.

Eig. vectors: $\text{null}(A - \lambda I) = \text{null}\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ +2

$\Rightarrow \vec{x}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ +2. Geometric multiplicity = 1.

$\text{null}\,(A - 2I)^2 = \text{null}\begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \dim(\text{null}(A-2I)^2) = 2.$ $\neq$ A.M

$\text{null}\,(A - 2I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \dim(\text{null}(A-2I)^3) = 3 = AM.$

+3

Choose $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to be Lin. indep from $\vec{v}_1$.

$$\vec{x}_2(t) = e^{2t}\left[\vec{v}_2 + t(A - \lambda I)\vec{v}_2 + \cancel{\frac{t^2}{2!}(A-\lambda I)^2\vec{v}_2 + \cdots}^{\,0}\right]$$

$$= e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] = e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$ +2

$$\vec{x}_3(t) = e^{2t}\left[\vec{v}_3 + t(A - \lambda I)\vec{v}_3 + \frac{t^2}{2!}(A - \lambda I)^2\vec{v}_3 + \cancel{\cdots}^{\,0}\right]$$

$$= e^{2t}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix} 6t + 5/2\,t^2 \\ 5t \\ 1 \end{pmatrix}$$ +2

Gen soln: $$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} 6t + \frac{5t^2}{2} \\ 5t \\ 1 \end{pmatrix}$$

+2

(15 points) *Problem 5.* Given an ODE system $\mathbf{x}'(t) = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$, find the general solution.

A possible differen solution

+2 pts eigenvalue $\lambda = 2$

+3 pts $A - 2I = \begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

+3 pts $(A-2I)^2 = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

+3 pts Formula $e^{At} = e^{2t}\left(I + (A-2I)t + \frac{(A-2I)^2}{2}t^2 + 0\right)$

Since $(A-2I)^k = 0$ for $k \geq 3$

$$e^{At} = e^{2t}\left(I + \begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{t^2}{2}\right)$$

+2 pts $$= e^{2t} \begin{pmatrix} 1 & t & 6t + \frac{5t^2}{2} \\ 0 & 1 & 5t \\ 0 & 0 & 1 \end{pmatrix}$$

+2 pts The general solution $X(t) = e^{At} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

(15 points) *Problem 6.* Given an ODE $x''(t) - 3x'(t) + 2x(t) = e^t$,

(i) Find a fundamental set of solutions of the corresponding homogeneous ODE.

(ii) Find one particular solution of the given ODE.

(iii) Find the general solution of the given ODE.

(6) i) $m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 2, 1 \Rightarrow \boxed{x_1(t) = e^{2t} \quad x_2(t) = e^t}$

roots +3
sol's +3

-1 for $c_1x_1 + c_2x_2$ as only answer

(6) ii) try $y_p = ae^t \Rightarrow$ Does not work (sol. to homog. eqn.)

try $y_p = ate^t$

$y_p' = ae^t + tate^t$

$y_p'' = ae^t + ae^t + ate^t$

$\Rightarrow -ae^t = e^t \Rightarrow a = -1$

$\boxed{y_p = -te^t}$

+2 try e^t
+2 try te^t
+2 final ans

(3) iii) $\boxed{y_g = -te^t + c_1e^{2t} + c_2e^t}$

+3 sum of ii + lin comb of i

error carried forward from ii allowed

Solutions

(15 points) *Problem 7.* Apply both Laplace and inverse Laplace transforms to solve the IVP: $y'' - 3y' - 4y = e^{-t}, y(0) = 0, y'(0) = -1$.

$$\mathcal{L}\{y'' - 3y' - 4y\} = \mathcal{L}\{e^{-t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) - 4Y(s) = \frac{1}{s+1}$$

$$s^2Y(s) + 1 - 3sY(s) - 4Y(s) = \frac{1}{s+1} \quad \boxed{+4} \quad (\text{1 pt each term})$$

$$\Rightarrow (s^2 - 3s - 4)Y(s) + 1 = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)(s^2-3s-4)} - \frac{1}{(s^2-3s-4)} \quad \boxed{+1} \left(\text{solve for } Y(s)\right)$$

$$= \frac{1}{(s+1)^2(s-4)} - \frac{1}{(s+1)(s-4)}$$

$$= \frac{-s}{(s+1)^2(s-4)}$$

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4} \quad \boxed{+2} \quad (\text{PF decomp})$$

$$= \frac{(s+1)(s-4)A + B(s-4) + C(s+1)^2}{(s+1)^2(s-4)}$$

$$\Rightarrow -s = A(s^2 - 3s - 4) + Bs - 4B + C(s^2 + 2s + 1)$$

$$\Rightarrow A + C = 0 \qquad -3A + B + 2C = -1 \qquad -4A - 4B + C = 0$$

$$C = -A \qquad -3A + B - 2A = -1 \qquad \Rightarrow -4A - 4(-1 + 5A) - A = 0$$

$$= -\frac{4}{25} \qquad -5A + B = -1 \qquad \Rightarrow A = \frac{4}{25}$$

$$-5A + B = -1$$

$$B = -1 + 5A$$

$$= -\frac{1}{5}$$

$$\Rightarrow Y(s) = \frac{4}{25}\frac{1}{s+1} - \frac{1}{5}\frac{1}{(s+1)^2} - \frac{4}{25}\frac{1}{s-4} \quad \boxed{+4} \quad (\text{putting in form to take ILT})$$

$$\Rightarrow y(t) = \underbrace{\frac{4}{25}e^{-t}}_{+1} \underbrace{-\frac{1}{5}te^{-t}}_{+2} \underbrace{-\frac{4}{25}e^{4t}}_{+1} \quad \boxed{+4}$$

Alternate P.F. decomp

(15 points) *Problem 7.* Apply both Laplace and inverse Laplace transforms to solve the IVP: $y'' - 3y' - 4y = e^{-t}, y(0) = 0, y'(0) = -1$.

$$Y(s) = \frac{-s}{(s+1)^2(s-4)}$$

$$= \frac{As+B}{(s+1)^2} + \frac{C}{s-4}$$

$$= \frac{(As+B)(s-4) + C(s+1)^2}{(s+1)^2(s-4)}$$

$$\Rightarrow -s = As^2 - 4As + Bs - 4B + C(s^2+2s+1)$$

$$\Rightarrow A + C = 0 \qquad -4A + B + 2C = -1 \qquad -4B + C = 0$$

$$C = -A \qquad\qquad -4B = -C$$

$$A = -C \qquad\qquad B = \frac{C}{4}$$

$$\Rightarrow 4C + \frac{C}{4} + 2C = -1$$

$$\frac{25C}{4} = -1$$

$$C = \frac{-4}{25}$$

$$\Rightarrow A = \frac{4}{25} \qquad B = \frac{-1}{25}$$

$$\Rightarrow Y(s) = \frac{4/25\, s - 1/25}{(s+1)^2} + \frac{(-4/25)}{s-4}$$

+3

$$= \frac{1}{25}\frac{(4s-1)}{(s+1)^2} - \frac{4}{25}\frac{1}{s-4}$$

$$= \frac{1}{25}\frac{4(s+1)}{(s+1)^2} - \frac{5}{25}\frac{1}{(s+1)^2} - \frac{4}{25}\frac{1}{s-4}$$

$$\Rightarrow y(t) = \frac{4}{25}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \frac{4}{25}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= \frac{4}{25}e^{-t} - \frac{1}{5}te^{-t} - \frac{4}{25}e^{4t}$$