COLOSTATE FALL 2013 MATH 340 EXAM 2

Thu. 11/14/2013

NAME: Answers CSUID: ______________

SECTION: ________

Problem	Score
1	10
2	15
3	15
4	15
5	15
6	15
7	15
Total	100

Exam Policy

(i) **No** calculator, textbook, homework, or notes should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use one 2-sided Cheat Sheet for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The three vector functions $\mathbf{u}_1(t) = [\cos(t), 0, -\sin(t)]^T$, $\mathbf{u}_2(t) = [\sin(t), 0, \cos(t)]^T$, $\mathbf{u}_3(t) = [0, e^t, 0]^T$ are linearly independent on the interval $(-\pi, \pi)$.

(ii) (T) (F) For the square matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\mathbf{v}_1 = [1, 0]^T$ is an eigenvector associated with $\lambda = 2$.

(iii) (T) (F) For the square matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\mathbf{v}_2 = [1, 1]^T$ is also an eigenvector associated with $\lambda = 2$.

(iv) (T) (F) Consider the ODE system $\begin{cases} x'(t) = 3x - 4y \\ y'(t) = 4x + 3y \end{cases}$. The equilibrium point at the origin is a spiral source.

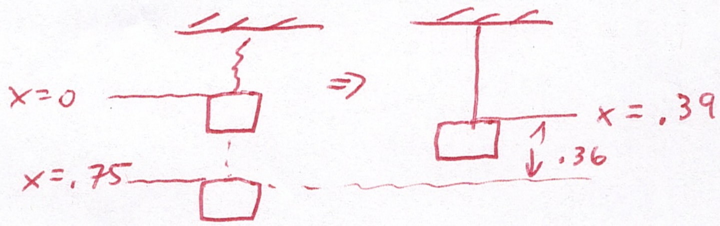
(v) (T) (F) For the square matrix $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, its exponential matrix $e^{At} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$.

+2 ea

(15 points) *Problem 2.* In an experiment,

- A 5-kg mass is attached to a spring.
- The displacement of the mass-spring equilibrium from the spring equilibrium is measured to be $0.75m$.
- The mass is then displaced $0.36m$ upward from the mass-spring equlibrium.
- Then the system is given a sharp downward tap, imparting an instantaneous downward velocity of $0.45m/s$.

Assume there is no damping present. Set up (but **do not solve**) an initial value problem for the resulting motion.



$$\boxed{\begin{aligned} &x'' + \frac{k}{m}x - g = 0 \\ &x(0) = .39m \\ &x'(0) = .45 \,m/s \end{aligned}}$$

}5 (-2 if no g)

}5

5

$kx = mg$

$$k = \frac{mg}{x} = \frac{5(9.8)}{.75} = \boxed{\frac{196}{3}\frac{kg}{s^2} = 65\tfrac{1}{3}\frac{kg}{s^2}}$$

(if "down" is $x<0$ then sign of both initial conditions and the "g" term in the diff eq. must change -2 for mis-match signs)

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w/ COV

$$y'' + \frac{k}{m}y = 0$$

$y(0) = -.36 \leftarrow$ -2 sign

$y'(0) = .45$

here $y = (x - x_0)$

(15 points) *Problem 3.* Consider an ODE system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}$.

10pts (i) Find a fundamental matrix $\mathbf{Y}(t)$.

5pts (ii) Find the inverse of $\mathbf{Y}(0)$.

(i) $A - \lambda I = \begin{bmatrix} -3-\lambda & -4 \\ 2 & 3-\lambda \end{bmatrix} \Rightarrow (-3-\lambda)(3-\lambda)+8=0$

$\Rightarrow \lambda^2 - 1 = (\lambda-1)(\lambda+1) = 0$

$\lambda_1 = 1, \quad \lambda_2 = -1$ 2pts

for $\lambda = 1$

3pts

$A - I = \begin{bmatrix} -4 & -4 \\ 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

−1 for incorrect neg sign (each)

for $\lambda = -1$

3pts

$A + I = \begin{bmatrix} -2 & -4 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$

$Y(t) = \begin{bmatrix} -e^{t} & -2e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}$ ← using 1st option

2pts

Then $Y(0) = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ and $Y(0)^{-1} = \frac{1}{+1}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

1pt

4pts = 1pt each entry

Other options:

For $Y(0) = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$, $Y(0)^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$

For $Y(0) = \begin{bmatrix} 2 & -1 \\ -1 & +1 \end{bmatrix}$, $Y(0)^{-1} = \begin{bmatrix} +1 & 1 \\ 1 & 2 \end{bmatrix}$

(15 points) *Problem 4.* Consider the autonomous ODE system $\begin{cases} x'(t) = -2y + x(x^2+4y^2-4) \\ y'(t) = \frac{1}{2}x - y(x^2+4y^2-4) \end{cases}$

5 pts (i) Verify that $x(t) = 2\cos(t), y(t) = \sin(t)$ is a solution.

5 pts { 1 pt (ii) – Is there any solution curve passing through the point $(x, y) = (1, 0)$? yes 1 pt

1 pt – How many? only one 1 pt

3 pts – Briefly justify your answer.

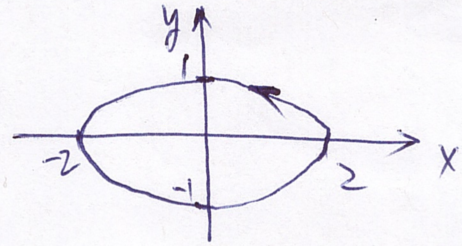
(iii) Assume that $(x(t), y(t))$ is a solution curve in part (ii).

5 pts { 2 pts – Can the solution curve go through the point $(x, y) = (3, 0)$?

3 pts – Briefly justify your answer.

(i) $\begin{cases} x'(t) = -2\sin t = -2y \\ y'(t) = \cos t = \frac{1}{2}x \end{cases}$ $\quad x^2+4y^2-4 = 4\cos^2 t + 4\sin^2 t - 4 \equiv 0$

So $\begin{cases} x(t) = 2\cos t \\ y(t) = \sin t \end{cases}$ is a solution



(ii) There is one and only one solution curve passing thru (1,0)

By the Existence & Uniqueness Thm. 1 pt

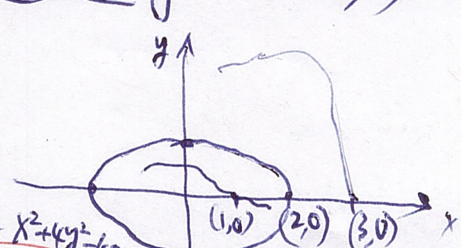
$$\left.\begin{aligned} f(x,y) &= -2y + x(x^2+4y^2-4) \\ g(x,y) &= \tfrac{1}{2}x - y(x^2+4y^2-4) \\ f_x &= (x^2+4y^2-4) + x\cdot 2x, \quad f_y = \cdots \\ g_x &= \tfrac{1}{2} - 2xy \qquad\qquad g_y = \cdots \end{aligned}\right\}$$ all are polynomials and hence continuous on $\mathbb{R}^2$ 2 pts

So the Existence & Uniqueness Thm holds.

(iii) the solution curve in part (ii) cannot go thru (3,0)

2 pts By the Uniqueness thm.

1 pt For this autonomous sys. two solution curves either are identical or never meet



2 pts { The solution curve in part (i) is the ellipse $x^2+4y^2-4=0$

The solution curve in part (ii) must stay inside the ellipse b/c (1,0) is so.

Thus the solution curve (ii) cannot go thru (3,0)

Otherwise, it crosses the solution curve in part (i)

(15 points) *Problem 5.* Find the solution to the initial value problem

$$y''(t) - 3y'(t) + 2y(t) = 0, y(0) = 2, y'(0) = -3.$$

$$\mu^2 - 3\mu + 2 = 0 \qquad \boxed{3}$$

$$\Rightarrow (\mu - 2)(\mu - 1) = 0 \qquad \boxed{3}$$

$$\Rightarrow \mu = 2 \text{ or } \mu = 1$$

General solution $y(t) = c_1 e^{2t} + c_2 e^{t}$ $\boxed{4}$

$$y'(t) = 2c_1 e^{2t} + c_2 e^{t} \qquad \boxed{1}$$

Initial conditions.

$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = 2c_1 + c_2 = -3 \qquad \boxed{4}$$

$$\Rightarrow c_1 = -5$$

$$c_2 = 7$$

$$\Rightarrow \boxed{y(t) = -5e^{2t} + 7e^{t}}$$

(15 points) *Problem 5.* Find the solution to the initial value problem

$$y''(t) - 3y'(t) + 2y(t) = 0, y(0) = 2, y'(0) = -3.$$

Method 2

$u_1 = y$
$u_2 = u_1'$ $\quad \Rightarrow \quad u_2' = 3u_2 - u_1$

$\boxed{2}$

$$\Rightarrow u' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Eigenvalues: $\boxed{3}$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} \right) = \lambda(\lambda - 3) + 2 = \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow \lambda_1 = 1 \text{ or } \lambda_2 = 2$$

Eigenvectors $\boxed{2}$

$\lambda_1 = 1,$

$$\text{null} \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$ $\boxed{2}$

$$\text{null} \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boxed{2}$$

Initial Conditions $\boxed{4}$

$$u(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \Rightarrow c_1 = 7, \ c_2 = -5$$

$$\Rightarrow u(t) = 7e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 5e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow y(t) = 7e^t - 5e^{2t}$$

(15 points) *Problem 6.* Consider an ODE system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$.

(i) Find its **real-valued** general solution;

(ii) Classify the equilibrium at the origin (stable/unstable? center/nodal/saddle/spiral?)

+11

① $\det\begin{pmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{pmatrix} = -\lambda(2-\lambda)+2 = 0$ $\qquad \dfrac{2 \pm \sqrt{4-8}}{2}$

$\lambda^2 - 2\lambda + 2 = 0$ (+2) $\qquad \boxed{\lambda = 1 \pm i}$ (+2)

$\lambda = 1+i$

$\vec{w} \cdot \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix} \longrightarrow x_2 = (1+i)x_1$ (+1) $\qquad \vec{w} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{v}_1} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{v}_2}$ (+4)

$\vec{x}(t) = C_1 e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right] + C_2 e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t\right]$ (+3)

+4

② source/ unstable (+2) spiral (+2)

$T = 2$, $D = 2$ $\qquad T^2 - 4D < 0 \rightarrow$ spirals

$T > 0 \rightarrow$ source/ unstable

Also acceptable:

$\vec{x}(t) = C_1 e^t\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right] + C_2 e^t\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t\right]$

$\vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\qquad$ ($\vec{v}_2$ can be off by a sign)

Fundamental set, no general solution: −2

Wrong eigenvalues, correct approach from there: +3 (half credit)

(15 points) *Problem 7.* Find the general solution of $\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$

$= (2-\lambda)(2-\lambda)(-1-\lambda)$ ← lower triangular matrix

$= -1(\lambda-2)^2(\lambda+1)$

$\Rightarrow \lambda = -1 \qquad \lambda = 2$ AM. 2 $\qquad \underline{+2}$

Find e-vectors

$\underline{\lambda=-1}$ $\quad A + I = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\left.\begin{matrix} v_1 = 0 \\ v_2 = 0 \\ v_3 = v_3 \end{matrix}\right\}$ $\quad \underline{+2}$

Choose $v_3 = -1$

$\vec{v}_{-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{y}_1(t) = e^{-t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\lambda = 2}$ $\quad A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} v_1 = 0 \\ v_3 = 0 \\ v_2 = v_2 \end{matrix}$ $\quad \underline{+2}$

Choose $v_2 = 1$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{y}_2(t) = e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Need additional soln:

$(A-2I)\vec{w} = \vec{v}_2$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} w_1 = 1 \\ w_3 = 0 \\ w_2 = w_2 \end{matrix}$ $\quad \underline{+2}$

Choose $w_2 = 1$ (or $w_2 = 0$)

$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{y}_3(t) = e^{2t}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$= e^{2t}\begin{bmatrix} 1 \\ 1+t \\ 0 \end{bmatrix}$

$\underline{+7}$ $\Rightarrow$ General soln

$\vec{y}(t) = \underbrace{c_1 e^{-t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{+2} + \underbrace{c_2 e^{2t}\begin{bmatrix} 1 \\ 1+t \\ 0 \end{bmatrix}}_{+2} + \underbrace{c_3 e^{2t}\begin{bmatrix} 1 \\ 1+t \\ 0 \end{bmatrix}}_{+2}$ $\quad \underline{+1}$ for general solution

Row-reducing before $\det(A-\lambda I)$ $\underline{-2}$

Error in finding e-vector (addition) $\underline{-1}$

Inconsistent system or $\vec{0}$ e-vector $\underline{-2}$

Incorrect form for 3rd soln ($\vec{w}te^{2t}$) $\underline{-1}$

Pick random vector for 3rd soln $\underline{-2}$

Alt. Soln

(15 points) *Problem 7.* Find the general solution of $\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

$$= (2-\lambda)^2(-1-\lambda)$$

$$= 0$$

+2

$$\Rightarrow \lambda_1 = 2 \quad \text{A.M. } 2 \qquad \lambda_2 = -1$$

$\lambda_1 = 2$ $\quad (A-2I)v = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \Rightarrow \begin{array}{l} v_1 = 0 \\ v_3 = 0 \end{array} \Rightarrow v_2 = \text{free} \qquad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad +2$$

$$\vec{y}_1(t) = e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find generalized e-vector

$$(A-2I)^2 v = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} \Rightarrow \begin{array}{l} v_3 = 0 \\ v_1 = \text{free} \end{array} \quad v_2 = \text{free} \qquad \vec{v}_2 = \begin{bmatrix} t \\ s \\ 0 \end{bmatrix}$$

$$\text{Pick } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left(\text{or } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) \quad +2$$

$$\Rightarrow \vec{y}_2(t) = e^{2t}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t\right\}$$

$$= e^{2t}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$\lambda_2 = -1$ $\quad (A+I)v = 0$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} v_1 = 0 \\ v_2 = 0 \\ v_3 = \text{free} \end{array} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad +2$$

$$\vec{y}_3(t) = e^{-t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Gen. Soln +1

$$\vec{y}(t) = \underbrace{C_1 e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{+2} + \underbrace{C_2 e^{2t}\left\{\begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}\right\}}_{+2} + \underbrace{C_3 e^{-t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{+2}$$