ColoState Fall 2013 Math 340 Exam 1

Thu. 10/10/2013

Name: Answers CSUID: 100

Section: ______

Problem	Score
1	10
2	15
3	15
4	15
5	15
6	15
7	15
Total	

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You could use one 2-sided Cheat Sheet for this exam.

Good luck!

(10 points) *Problem 1.* True or False, circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The constant functions $x(t) = k\pi$ (k any integer) are all solutions of the ODE $x'(t) = \sin(x)\ln(t)$.

(ii) (T) (F) The function $x(t) = t^2 + \dfrac{1}{t^2}$ (for $t > 0$) is a solution to the initial value problem $tx' + 2x = 4t^2, x(1) = 1$.

(iii) (T) (F) The ODE $\sin(y)dx + (1 - x\sin(y))dy = 0$ is exact.

(iv) (T) (F) The two vectors $\mathbf{u} = (1, 2, -5), \mathbf{v} = (-1, -2, 3)$ are linearly independent.

(v) (T) (F) If $\mathbf{x}_1, \mathbf{x}_2$ are both solutions of a nonhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

(i) just plug-in
the derivative of a const. fxn. is zero
$\sin(k\pi) = 0$ for any integer k.

(ii) The fxn. $x(t) = t^2 + \frac{1}{t^2}$ satisfies the ODE
but not the initial condition (IC)

(iii) $P = \sin(y)$, $\quad \partial_y P = \cos y$
$Q = 1 - x\sin(y)$, $\quad \partial_x Q = -\sin y$
not equal

(iv) Two vectors of the same size are lin. dep.
iff one is a multiple of the other

(v) $Ax_1 = b$
$Ax_2 = b$
$\xrightarrow{\text{subtraction}}$
$Ax_1 - Ax_2 = 0$
$A(x_1 - x_2) = 0$

(15 points) *Problem 2.* Find the solution of $y'(x) = 2\cos(x)\sqrt{y}$ that satisfies the initial condition $y(\pi/4) = 2$.

$$\frac{dy}{dx} = 2\cos(x)\sqrt{y}$$

/3 ~ $$\int \frac{1}{\sqrt{y}}\,dy = \int 2\cos(x)\,dx$$

/2 /1

/2 ~ $$2\sqrt{y} = 2\sin(x) + C$$

$$\sqrt{y} = \sin(x) + C$$

$$y(x) = (\sin(x) + C)^2$$ /3

/2 ~ $$2 = y\left(\tfrac{\pi}{4}\right) = \left(\sin\left(\tfrac{\pi}{4}\right) + C\right)^2$$

$$\sqrt{2} = \pm\left(\frac{\sqrt{2}}{2} + C\right)$$

/2 ~ $C = \frac{\sqrt{2}}{2}$ or $\underbrace{C = -\frac{3\sqrt{2}}{2}}$

does not solve DE
(plug in C, x, y into DE to see this

$$\boxed{y(x) = \left(\sin(x) + \frac{\sqrt{2}}{2}\right)^2}$$

- if don't solve for y: -2
- if distribute ²: -3

(15 points) *Problem 3.* Given a linear ODE $y'(x) = y\tan(x) + \sin(x)$,

(i) Find the general solution.

(ii) Find the particular solution satisfying $y(0) = 2$.

i) int. factor: $u = e^{-\int \tan(x)dx}$

+12

$= e^{-[-\ln|\cos(x)|]} = |\cos(x)|$

(+4) $\longrightarrow$ use $\boxed{u = \cos(x)}$

$[\cos(x) \cdot y]' = \cos(x) \cdot \sin(x)$

(+4) $\longrightarrow$ $\cos(x) \cdot y = \frac{1}{2}\sin^2(x) + C$

(+4) $\longrightarrow$ $\boxed{y(x) = \frac{1}{2}\frac{\sin^2(x)}{\cos(x)} + \frac{C}{\cos(x)}}$

ii) $y(0) = 2 \Rightarrow C = 2$

+3

$\boxed{y(x) = \frac{1}{2}\frac{\sin^2(x)}{\cos(x)} + \frac{2}{\cos(x)}}$

(15 points) $\boxed{\textit{Problem 4.}}$ It is known that the ODE $\underbrace{2y}_{P}dx + \underbrace{(x+\sqrt{y})}_{Q}dy = 0$ is not exact but has an integrating factor in the form $\mu(y)$.

9pts (i) Find the integrating factor $\mu(y)$.

6pts (ii) Use the integrating factor to find a general solution of the ODE.

(i) $\mu(y) = e^{-\int g(y)dy}$ where $g(y) = \frac{1}{P}(P_y - Q_x) = \underbrace{\frac{(2-1)}{2y} = \frac{1}{2y}}_{3\text{pts}}$

$= \underbrace{e^{-\frac{1}{2}\int \frac{1}{y}dy}}_{2\text{pts}} = \underbrace{e^{-\frac{1}{2}\ln|y|}}_{2\text{pts}} = \underbrace{\frac{1}{\sqrt{y}}}_{2\text{pts}}$

Check: $2\sqrt{y}dx + (\frac{x}{\sqrt{y}} + 1)dy = 0$

$P_y = \frac{2}{2}\frac{1}{\sqrt{y}}$ $\quad Q_x = \frac{1}{\sqrt{y}}$ $\quad P_y = Q_x$ ✓

(ii) way 1

$F(x,y) = \int \underbrace{2\sqrt{y}dx}_{1\text{pt}} = 2x\sqrt{y} + \varphi(y)$ 1pt

$\frac{\partial F}{\partial y} = \frac{x}{\sqrt{y}} + \varphi'(y) = \frac{x}{\sqrt{y}} + 1 \quad \Rightarrow \int \varphi'(y) = \int 1 dy$

↑ 1pt ↑ 1pt

$\underbrace{\varphi(y) = y}_{1\text{pt}}$

way 2

$F(x,y) = \int \underbrace{\frac{x}{\sqrt{y}} + 1\, dy}_{1\text{pt}} = 2x\sqrt{y} + y + \varphi(x)$ 1pt

$\frac{\partial F}{\partial x} = 2\sqrt{y} + \varphi'(x) = 2\sqrt{y} \quad \Rightarrow \int \varphi'(x) = \int 0 dx$

↑ 1pt ↑ 1pt

$\underbrace{\varphi(x) = c}_{1\text{pt}}$

$\therefore \underbrace{F(x,y) = 2x\sqrt{y} + y = c}_{1\text{pt}}$

(15 points) *Problem 5.*

8pts (i) Show that the ODE $x'(t) = e^{-t^2}\sqrt[3]{x}$ has a unique solution satisfying the initial condition $x(1) = 1$.

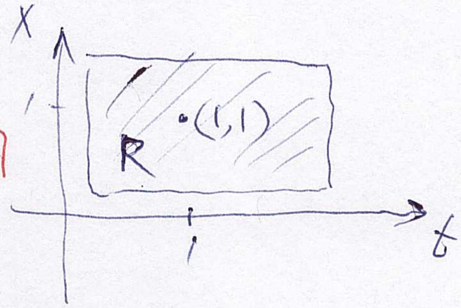
7pts (ii) Consider another initial value problem (IVP): $x'(t) = e^{-t^2}\sqrt[3]{x}$, $x(1) = 0$. Does the Uniqueness Theorem guarantee a unique solution for this IVP? Justify your answer.

$R = \{(t,x) : -\infty < t < +\infty,\ 0 < x < +\infty\}$

(i) 8pts ~~Consider~~ There exists a rectangular domain $R \ni (1,1)$ $\boxed{2\text{pts}}$

$f(t,x) = e^{-t^2}\sqrt[3]{x}$

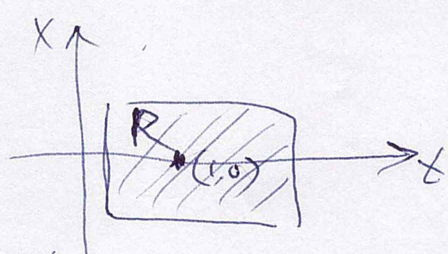
$\partial_x f = e^{-t^2}\frac{1}{3}x^{-\frac{2}{3}} = e^{-t^2}\frac{1}{3\sqrt[3]{x^2}}$ $\boxed{2\text{pts}}$



are both continuous on R $\boxed{2\text{pts}}$

So we can apply the Uniqueness Thm. $\boxed{2\text{pts}}$ to obtain a unique solution for this IVP

(ii) 7pts Any rectangular domain R ~~that~~ contains the point $(1,0)$ must contain a piece of the t-axis on which $x=0$



But $\partial_x f = e^{-t^2}\frac{1}{3\sqrt[3]{x^2}}$ is discontinuous at $(1,0)$ and any point for which $x=0$. $\boxed{4\text{pts}}$

So we cannot apply the Uniqueness Thm.

The Uniqueness Thm does not guarantee a unique solution for this IVP $\boxed{3\text{pts}}$

(15 points) *Problem 6.* Consider a given autonomous ODE $x'(t) = (x+1)(x-4)$.

(i) Find all equilibrium points.

(ii) Classify each equilibrium point as either unstable or asymptotically stable.

(iii) Sketch the equilibrium solutions in the tx-plane. These equilibrium solutions divide the tx-plane into regions. Sketch at least one solution trajectory in each of these regions.

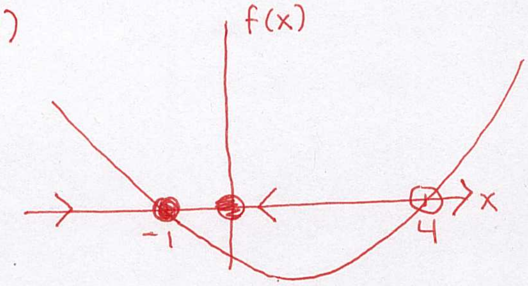
(i) $x'(t) = (x+1)(x-4) = f(x)$

Eq. pts $f(x) = (x+1)(x-4) = 0$

$\Rightarrow x = -1, \ x = 4$

$\boxed{\text{4pts - 2 pts each pt}}$

(ii)



$f(-2) > 0$

$f(0) < 0$

$f(5) > 0$

$\boxed{\text{5 pts; -2 pts method/plot; -3 pts conclusion}}$

$x = -1$ asymptotically stable

$x = 4$ unstable

Alt. 1st deriv test

$f'(x) = 2x - 3$

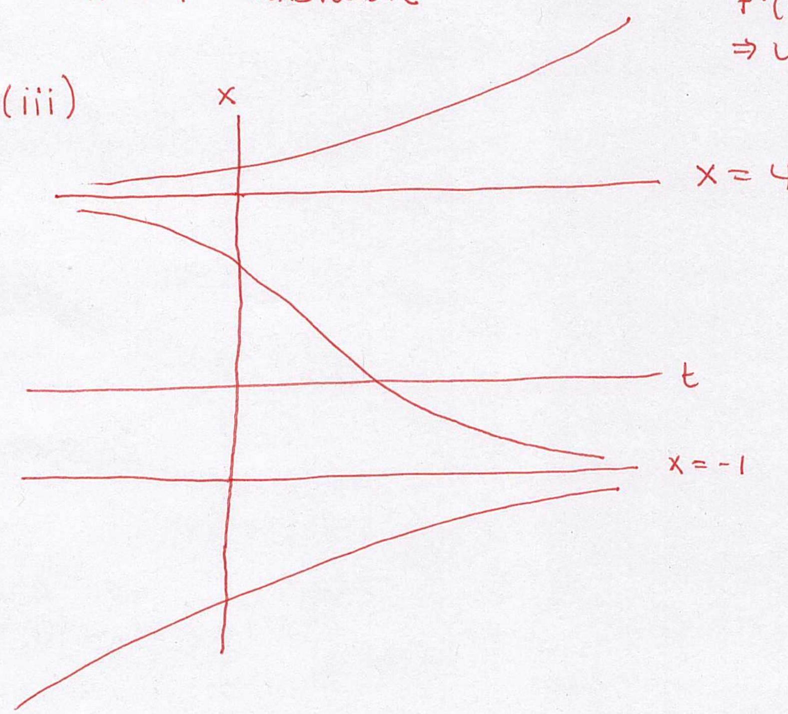
$f'(-1) = -5 < 0$

$\Rightarrow$ asymp. stable

$f'(4) = 5 > 0$

$\Rightarrow$ unstable

(iii)



$\boxed{\text{6 pts; 2 pts for each region}}$

(15 points) *Problem 7.* Consider a given matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{bmatrix}$.

(i) Apply elementary row operations to simplify the matrix to the **row echelon form.**

(ii) Write the solutions of the linear system $\mathbf{Ax} = \mathbf{0}$ in the parametric form.

(iii) Find a basis for null($\mathbf{A}$).

(5)

(i) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix} \xrightarrow{R_3 + R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(5)

(ii) $A\vec{x} = \vec{0} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Set $x_3 = t$. $x_1 = -t$, $x_2 = 0$, $x_3 = t$

Free column

$\Rightarrow$ In parametric form, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t$

(5)

(iii) basis of null(A) $= \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$