Tools To Tame Tensors

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This work was partially supported by NSF grant DMS-1620454.

joint with Uriyah First, Joshua Malgione

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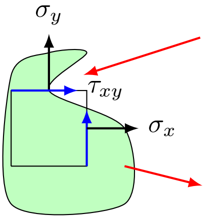
Meet the rest of the team

Uriyah First
U. British Columbia
Categories and Schemes

Joshua Maglione
Colorado State University
Calculations and Software

Tensors are very general objects

$$dx_1 \wedge \cdots \wedge dx_s$$



$$\frac{1}{\sqrt{2}}\langle 00| + \langle 11|$$

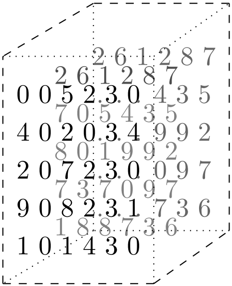
$$R^{j_1 \dots j_t}_{i_1 \dots i_s}$$

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\{abc\} = (U_{a+c} - U_a - U_c)(b)$$

r	c	v
1	5	0.7
101	12	−1.1
8	50	−9
⋮	⋮	⋮



$$[x,y] = xy - yx$$

$$A \otimes B$$

$$K_*(F) = T^*F/(a \otimes (1-a))$$

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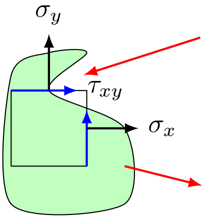
Gauss–Ricci interpretation

- $dx_i \wedge \cdots \wedge dx_s$ bases for algebra generated by directional derivatives.
- Christoffel symbols Γ^k_{ij} and Ricci tensors $R^{j_1 \cdots j_t}_{i_1 \cdots i_s}$ are coefficients of linear combinations.
- Invariants, e.g. curvature, is the evaluation tensors such as the Ricci tensor R. Levi-Civita connection ∇ smoothly moves one tangent algebra to the next.

Tensors are very general objects

Hamilton & Copenhagen interpretation

- Kinematics driven by multiple input vectors, e.g. the *stress tensor* on an object, i.e. $\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$.
- Quantum k-state particle modeled by $\mathbb{C}^k$ it's states $\{\langle i| : i \in \{1, \ldots, k\}\}$ a basis.
- Entanglment of states $\langle \psi| \in \mathbb{C}^a$ with $\langle \tau| \in \mathbb{C}^b$ is non-pure tensor in $\mathbb{C}^a \otimes \mathbb{C}^b$, simplest example the Bell pair $\frac{1}{\sqrt{2}}\langle 00| + \langle 11|$



$\frac{1}{\sqrt{2}}\langle 00| + \langle 11|$

Hamilton invented the word "tensor" to mean the real part of a quaternion. Translation of 3-dimensional mechanics to quaternions lead to adopting the term universally.

Tensors are very general objects

Whitney interpretation

- tensor = multilinear
- hom(A, B) (matrices) a **space of tensors**, i.e. its elements are tensors.
- $A \otimes B$ **space of cotensors**, i.e. its quotients are tensors.
- iterated these become interesting, e.g hom$(A \otimes B, C) \cong$ hom$(A, \text{hom}(B, C))$, T^*F, $\wedge^n V$, & $K_*(F)$.

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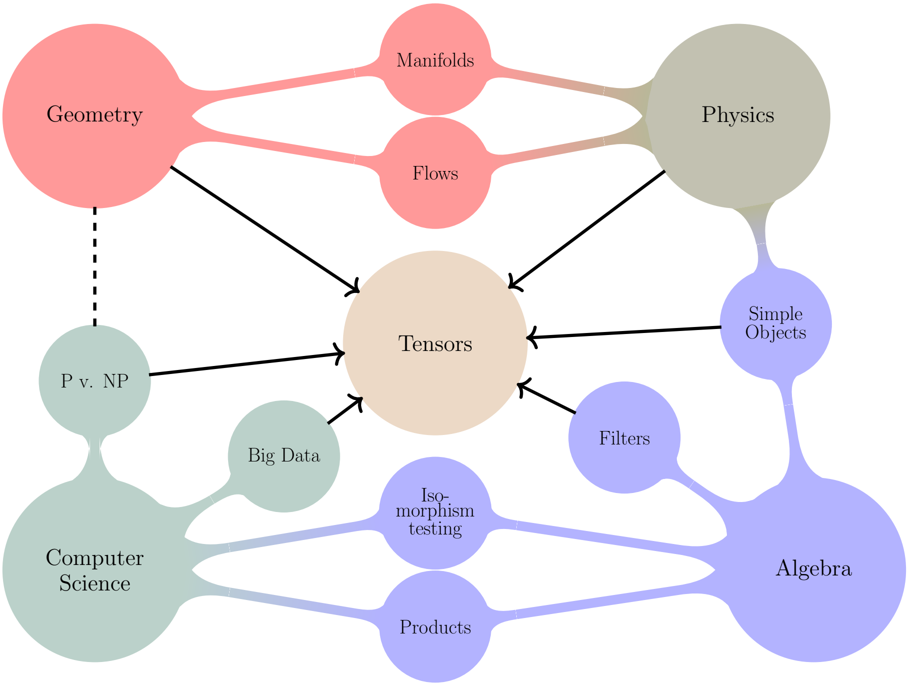
(Big) Data interpretation

- Many data are collected through time (video) or space (MRI), or coded by value vectors (PageRank).
- Modeling data as “volumes” allows comparison along time, space, and coding entries.
- This makes data into natural tensors.
- But in practice we sacrifice volumes for sparse representations.


Geometry
Manifolds
Flows
Physics
Tensors
Simple Objects
P v. NP
Big Data
Filters
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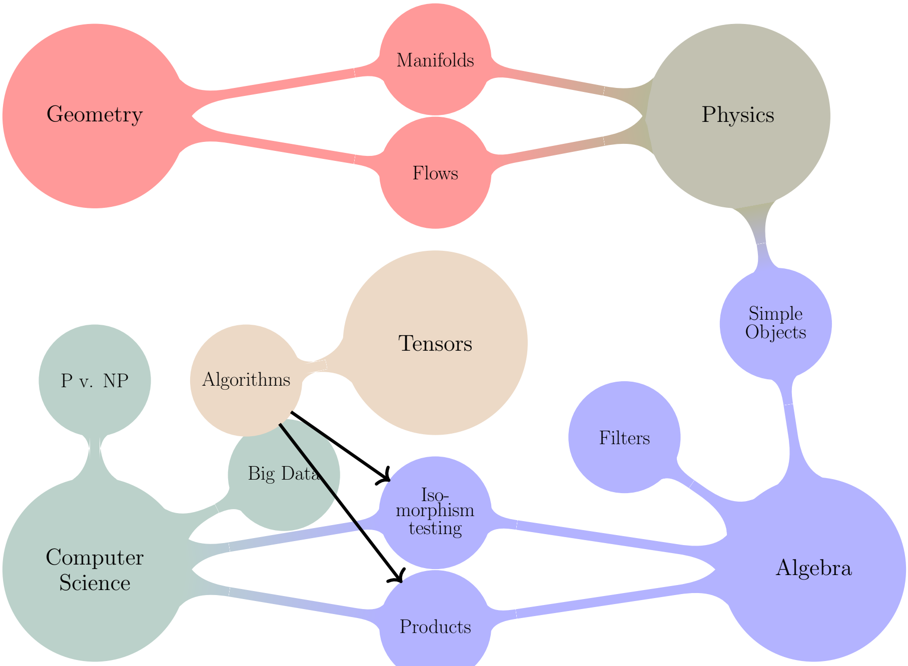
Products
Computer Science
Algebra

Tensors Uncover Algorithms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Decomposition Algorithms

For decades decomposing groups as $G = H \times K$ took testing **every** subgroup, so $\exp(O(\log^2 |G|))$-steps.

Then non-associative algebra stepped in.

THM. (W. 2008)

There is an algorithm to construct a direct product decomposition of a finite groups G in time $O(\log^7 |G|)$.
In fact true for much wilder central products as well.

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E.g. with central products

Pre-Jordan Algebra techniques

- Central products had no Krull-Schmidt:, e.g. $D_8 \circ D_8 \cong Q_8 \circ Q_8$; also Tang: $\exists T, R$ centrally indecomposable with $R \circ R \circ R \cong T \circ R$.
- Almost all theorems required groups with cyclic center.

Post-Jordan Algebra viewpoint

- Instead of Krull-Schmidt, Jordan algebra classify all **orbits** of central product decompositions.
- Tang's becomes natrual: symmetric forms in char 2 are also alternating. I.e. **group analogue to well-known topology rules:** $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \cong \mathbb{T}^2 \# \mathbb{RP}^2$.

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Filter Refinements

Filter

A **filter** $\phi : M \to 2^G$ from a commutative preordered monoid M into subgroups of G satisfies

$$[\phi_s, \phi_t] \leq \phi_{s+t} \qquad\qquad s \prec t \Rightarrow \phi_s \geq \phi_t.$$

Write $G = \varprojlim \phi_s$.

Filters give graded algebras

$[x, y]$ in G factors through a grade Lie algebra product on:

$$L(\phi) = \bigoplus_{s \neq 0} \phi_s / \partial\phi_s \qquad\qquad \partial\phi_s = \langle \phi_{s+t} : t \neq 0 \rangle.$$

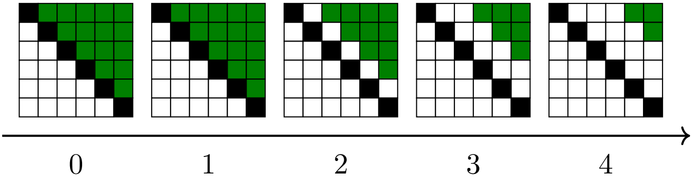
*Ascending version, e.g. upper central series, gives graded module.

Thm W.

Fix a filter $\phi : M \to 2^G$ and $X \leq G$ such that $\exists s$, $\partial\phi_s X \leq \phi_s$. Then $\exists\widehat{\phi} : M \times \mathbb{N} \to 2^G$ refining ϕ to include X. That is:

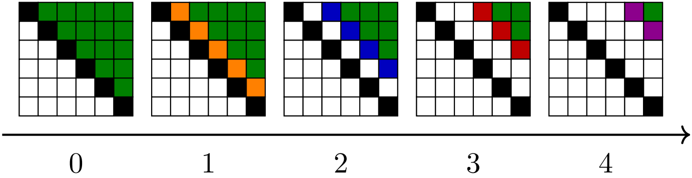
$$G = \varprojlim \phi_s = \varprojlim_{s \in M} \varprojlim_{t \in \mathbb{N}} \widehat{\phi}_{(s,t)}.$$

Notice refinement is recursive.



J. Maglione (2017)

$$L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K$$



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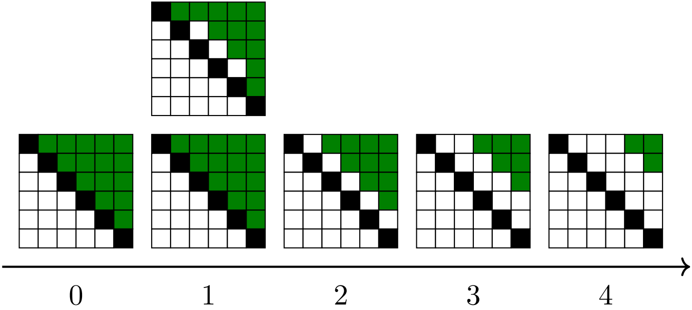

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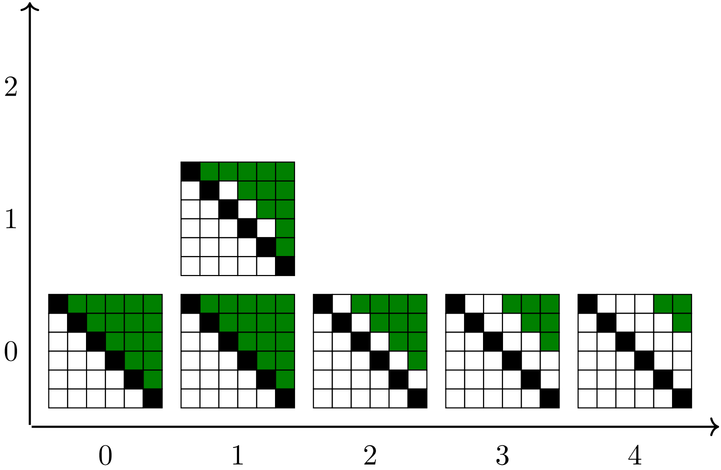

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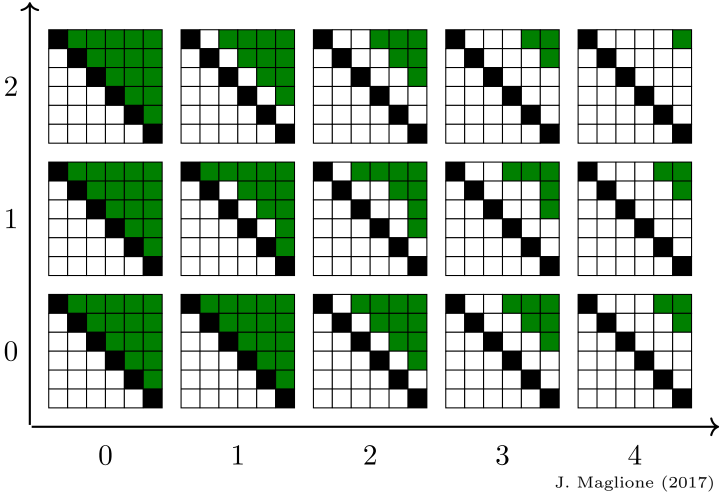
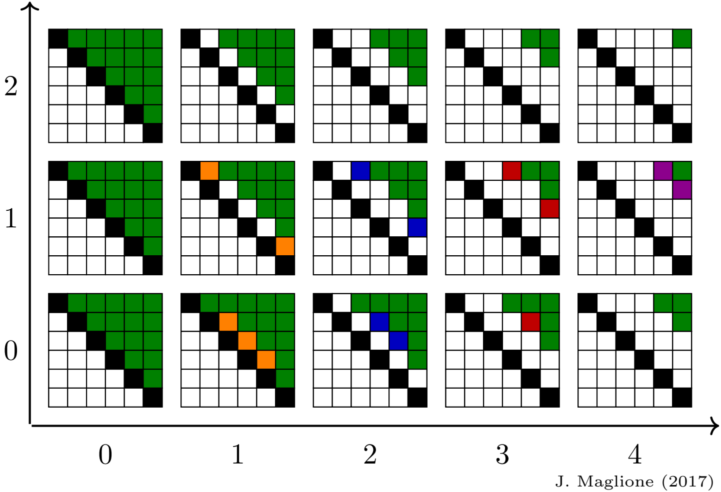


$$L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K$$

$$L(\phi) = K^3 \oplus K^2 \oplus K^2 \oplus K^2 \oplus K \oplus K^2 \oplus K^2 \oplus K$$



Application

Thm. (W.)

On a log-scale a positive proportion of all nilpotent groups admits a proper characteristic refinement of the lower exponent p central series.

Maglione-W.

A survey of 500,000,000 random class 2 groups found 97% refined, 92% to maximal class!

Already applied to improve isomorphism testing exponentially at random.

A Category for Tensor Spaces

$$V_{a+1} \oslash \cdots \oslash V_0 = \hom(V_{a+1}, \hom(\ldots, \hom(V_1, V_0) \cdots).$$

DEF.

A **tensor space** is a K-module T and a monomorphism

$$|\cdot\rangle : T \hookrightarrow V_{\vav} \oslash \cdots \oslash V_0.$$

Elements of T are **tensors**, $\{V_{\vav}, \ldots, V_0\}$ is the **frame**, $\vav + 1$ **valence**.

Fix $\langle v| = \langle v_1| \cdots \langle v_{\vav}| \in V_1 \times \cdots \times V_{\vav}$

$$\langle v|t\rangle = \langle v_1| \cdots \langle v_{\vav}|t\rangle \in V_0.$$

So $|t\rangle : V_1 \times \cdots \times V_{\vav} \rightarrowtail V_0$ is K-multilinear; yet, t is anything.

Tensor categories

Pretend tensors are nonassociative algebras...

$$\boxed{A \xrightarrow{\phi} B}$$

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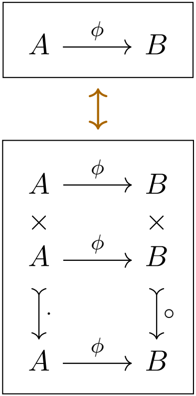

$$(a_2 \cdot a_1)\phi = a_2\phi \circ a_1\phi$$

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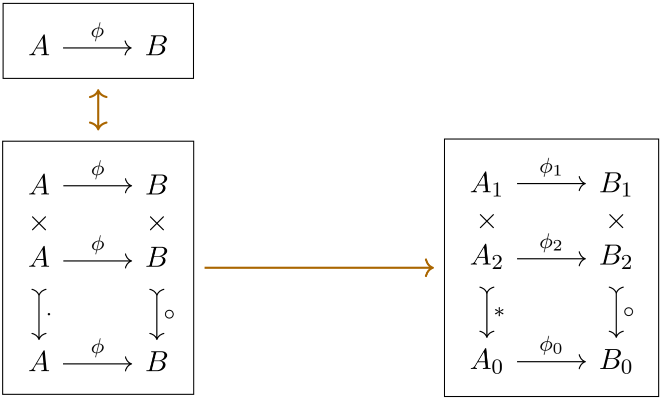

$(a_2 \cdot a_1)\phi = a_2\phi \circ a_1\phi$ $(a_2 \cdot a_1)\phi = a_2\phi \circ a_1\phi$

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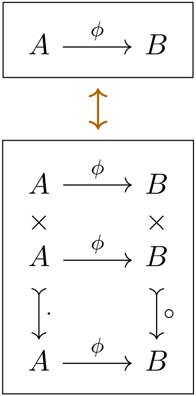

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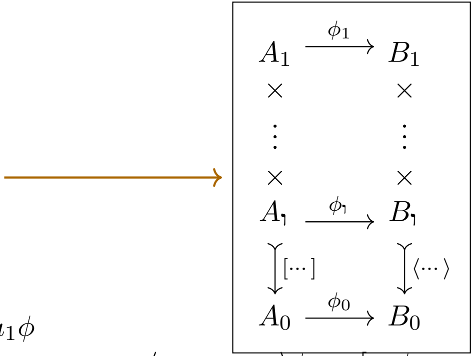

$$\langle a_v, \ldots, a_1 \rangle \phi_0 = [a_v\phi_v, \ldots, a_1\phi_1]$$

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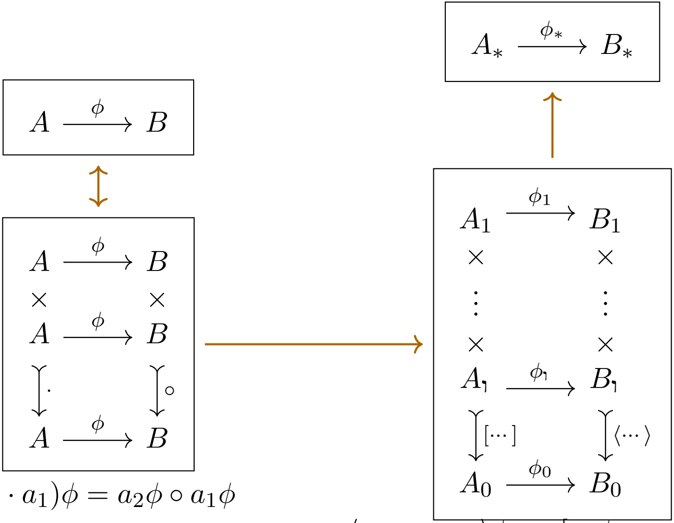

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$$\langle a_v, \ldots, a_1 \rangle \phi_0 = [a_v\phi_v, \ldots, a_1\phi_1]$$

Composition

$$A_* \xrightarrow{\phi_*} B_* \xrightarrow{\tau_*} C_* \quad\longrightarrow\quad A_* \xrightarrow{\phi_* \tau_*} C*$$

$$\begin{array}{ccccc} A_1 & \xrightarrow{\phi_1} & B_1 & \xrightarrow{\tau_1} & C_1 \\ \times & & \times & & \times \\ \vdots & & \vdots & & \vdots \\ \times & & \times & & \times \\ A_\daleth & \xrightarrow{\phi_\daleth} & B_\daleth & \xrightarrow{\tau_\daleth} & C_\daleth \\ \downarrow & & \downarrow & & \downarrow \\ A_0 & \xrightarrow{\phi_0} & B_0 & \xrightarrow{\tau_0} & C_0 \end{array} \quad\longrightarrow\quad \begin{array}{ccc} A_1 & \xrightarrow{\phi_1 \tau_1} & C_1 \\ \times & & \times \\ \vdots & & \vdots \\ \times & & \times \\ A_\daleth & \xrightarrow{\phi_\daleth \tau_\daleth} & C_\daleth \\ \downarrow & & \downarrow \\ A_0 & \xrightarrow{\phi_0 \tau_0} & C_0 \end{array}$$

Need even more morphisms

Can't remove triviality

$$\begin{array}{ccc} A_1 & \longrightarrow & A_1/A_2^{\perp} \\ \times & & \times \\ A_2 & \longrightarrow & A_2/A_1^{\perp} \\ \Big\downarrow{\scriptstyle *} & & \Big\downarrow{\scriptstyle \circ} \\ A_0 & \hookleftarrow & A_1 * A_2 \end{array}$$

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Obvious abelian category

$$\begin{array}{ccc} A_1 & \xrightarrow{\phi_1} & B_1 \\ \times & & \times \\ A_2 & = & B_2 \\ \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle \circ} \\ A_0 & \xrightarrow{\phi_0} & B \end{array}$$

$$(a_1 * a_2)\phi_0 = a_1\phi_1 \circ a_2$$

Need even more morphisms

Can't remove triviality

$$\begin{array}{ccc} A_1 & \longrightarrow & A_1/A_2^{\perp} \\ \times & & \times \\ A_2 & \longrightarrow & A_2/A_1^{\perp} \\ \downarrow{*} & & \downarrow{\circ} \\ A_0 & \hookleftarrow & A_1 * A_2 \end{array}$$

Non-abelian category!

$$\begin{array}{ccc} A_1 & \xrightarrow{\phi_2} & B_1 \\ \times & & \times \\ A_2 & \xrightarrow{\phi_1} & B_2 \\ \downarrow{*} & & \downarrow{\circ} \\ A_0 & = & B_0 \end{array}$$

$$(a_1 * a_2) = a_1\phi_1 \circ a_2\phi_2$$

Need even more morphisms

Can't remove triviality

$$\begin{array}{ccc} A_1 & \longrightarrow & A_1/A_2^{\perp} \\ \times & & \times \\ A_2 & \longrightarrow & A_2/A_1^{\perp} \\ \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle \circ} \\ A_0 & \hookleftarrow & A_1 * A_2 \end{array}$$

Now re-abelianized!

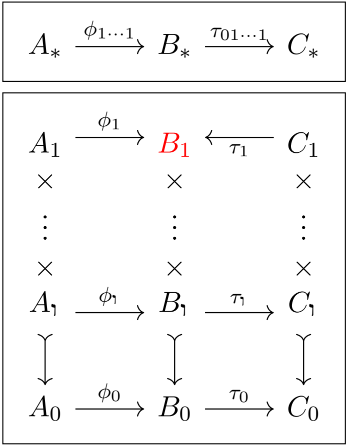
$$\begin{array}{ccc} A_1 & \xleftarrow[\phi_1]{} & B_1 \\ \times & & \times \\ A_2 & \xrightarrow{\phi_2} & B_2 \\ \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle \circ} \\ A_0 & = & B \end{array}$$

$$(b_1 \phi_2 * a_2) = b_1 \circ \phi_2 a_2$$

More Morphisms ⇒ Composition Issues

$$A_* \xrightarrow{\phi_{1\cdots 1}} B_* \xrightarrow{\tau_{01\cdots 1}} C_*$$

$$\begin{array}{ccccc}
A_1 & \xrightarrow{\phi_1} & \color{red}{B_1} & \xleftarrow[\tau_1]{} & C_1 \\
\times & & \times & & \times \\
\vdots & & \vdots & & \vdots \\
\times & & \times & & \times \\
A_\urcorner & \xrightarrow{\phi_\urcorner} & B_\urcorner & \xrightarrow{\tau_\urcorner} & C_\urcorner \\
\downarrow & & \downarrow & & \downarrow \\
A_0 & \xrightarrow{\phi_0} & B_0 & \xrightarrow{\tau_0} & C_0
\end{array}$$



Compose as relations.

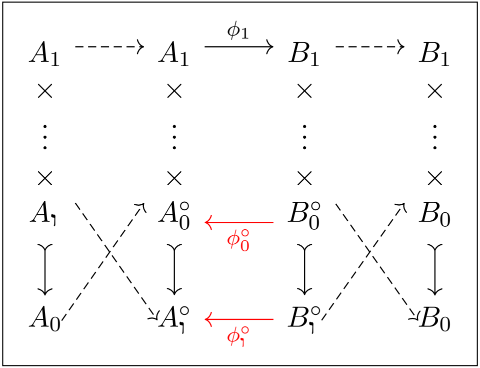
$$\phi_1 = \{(a, a\phi_1) : a \in A_1\}$$
$$\tau_1 = \{(c\tau_1, c) : c \in C_1\}$$

Define

$$\phi_1\tau_1 = \{(a, c) : \exists b, (a, b) \in \phi_1, (b, c) \in \tau_1\}$$

Works the same no matter direction of arrows.

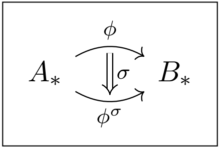
$$A_* \overset{\sigma}{\dashrightarrow} A_*^{\sigma} \xrightarrow{\phi_*} B_*^{\sigma} \overset{\sigma^{-1}}{\dashrightarrow} B_*$$



In Ricci calculus: "raising" and "lowering" indices.

In algebra: Knuth-Liebler transposes.

In our model: permutations σ of the frame give 2-morphisms



The 2-category of tensor spaces

Category= Objects + hom-**sets** (with some rules)
2-Category= Objects + hom-**categories** (with more rules)

$\vav$-Tensor space 2-cateogry

Objects Tensor spaces $|\cdot\rangle : T \hookrightarrow V_{\vav} \oslash \cdots \oslash V_0$ of valence $\vav + 1$.

1-Morphisms Linear **relations** $(F_{\vav}, \ldots, F_0)$ where

2-Morphisms Frame Braiding

We now have: subtensors, ideals, quotients, kernels, image, Noether's isomorphism theorems, products, coproducts, simples, projectives, representations, modules, ...

Tensor's can have modules

E.g.: Representations and modules of tensors

$$\boxed{A \xrightarrow{\rho} \mathrm{End}(M)}$$

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$$\Updownarrow$$

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \mathrm{End}(M) \\ \times & & \times \\ A & \xrightarrow{\rho} & \mathrm{End}(M) \\ \big\downarrow \cdot & & \big\downarrow \circ \\ A & \xrightarrow{\rho} & \mathrm{End}(M) \end{array}$$

$$(a_2 \cdot a_1)\rho = a_2\rho \circ a_1\rho$$

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$$(a_2 \cdot a_1)\rho = a_2\rho \circ a_1\rho \qquad (a_2 * a_1)\rho_0 = a_2\rho_2 \circ a_1\rho_1$$

E.g.: Representations and modules of tensors

$$A \xrightarrow{\rho} \mathrm{End}(M) \qquad\qquad A_* \xrightarrow{\rho_*} \hom(M_*)$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \mathrm{End}(M) \\ \times & & \times \\ A & \xrightarrow{\rho} & \mathrm{End}(M) \\ \downarrow \cdot & & \downarrow \circ \\ A & \xrightarrow{\rho} & \mathrm{End}(M) \end{array} \quad\longrightarrow\quad \begin{array}{ccc} A_2 & \xrightarrow{\rho_2} & \hom(M_2, M_1) \\ \times & & \times \\ A_1 & \xrightarrow{\rho_1} & \hom(M_1, M_0) \\ \downarrow * & & \downarrow \circ \\ A_0 & \xrightarrow{\rho_0} & \hom(M_2, M_0) \end{array}$$

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E.g.: Representations and modules of tensors

Representation

$$A_* \xrightarrow{\rho_*} \hom(M_*)$$

$$\updownarrow$$

$$\begin{array}{ccc}
A_2 & \xrightarrow{\rho_2} & \hom(M_2, M_1) \\
\times & & \times \\
A_1 & \xrightarrow{\rho_1} & \hom(M_1, M_0) \\
\downarrow * & & \downarrow \circ \\
A_0 & \xrightarrow{\rho_0} & \hom(M_2, M_0)
\end{array}$$

$$(a_2 * a_1)\rho_0 = a_2\rho_2 \circ a_1\rho_1$$

E.g.: Representations and modules of tensors

Right Representation

$$A_* \xrightarrow{\rho_*} \obslash(M_*)$$

$$\Updownarrow$$

$$\begin{array}{ccc}
A_2 & \xrightarrow{\rho_2} & M_2 \obslash M_1 \\
\times & & \times \\
A_1 & \xrightarrow{\rho_1} & M_1 \obslash M_0 \\
\big\downarrow * & & \big\downarrow \circ \\
A_0 & \xrightarrow{\rho_0} & M_2 \obslash M_0
\end{array}$$

$$(a_2 * a_1)\rho_0 = a_2\rho_2 \circ a_1\rho_1$$

E.g.: Representations and modules of tensors

Right Representation

$$A_* \xrightarrow{\rho_*} \oslash(M_*)$$

$\updownarrow$

$$\begin{array}{ccc} M_2 \times A_2 & \xrightarrow{\succ} & M_1 \\ M_1 \times A_1 & \xrightarrow{\prec} & M_0 \\ M_2 \times A_0 & \xrightarrow{\curlyvee} & M_0 \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} A_2 & \xrightarrow{\rho_2} & M_2 \oslash M_1 \\ \times & & \times \\ A_1 & \xrightarrow{\rho_1} & M_1 \oslash M_0 \\ \downarrow * & & \downarrow \circ \\ A_0 & \xrightarrow{\rho_0} & M_2 \oslash M_0 \end{array}$$

$(m_2 \succ a_2) \prec a_1 = m_2 \curlyvee (a_2 * a_1)$

$(a_2 * a_1)\rho_0 = a_2\rho_2 \circ a_1\rho_1$

E.g.: Representations and modules of tensors

Right Triptych

$$M_* \times A_* \longrightarrow M_*$$

$$\updownarrow$$

$$\begin{array}{ccc} M_2 \times A_2 & \overset{\succ}{\rightarrowtail} & M_1 \\ M_1 \times A_1 & \overset{\prec}{\rightarrowtail} & M_0 \\ M_2 \times A_0 & \overset{\curlyvee}{\rightarrowtail} & M_0 \end{array}$$

$$(m_2 \succ a_2) \prec a_1 = m_2 \curlyvee (a_2 * a_1)$$

$$\longleftrightarrow$$

Right Representation

$$A_* \xrightarrow{\rho_*} \obslash(M_*)$$

$$\updownarrow$$

$$\begin{array}{ccc} A_2 & \xrightarrow{\rho_2} & M_2 \obslash M_1 \\ \times & & \times \\ A_1 & \xrightarrow{\rho_1} & M_1 \obslash M_0 \\ \downarrow * & & \downarrow \circ \\ A_0 & \xrightarrow{\rho_0} & M_2 \obslash M_0 \end{array}$$

$$(a_2 * a_1)\rho_0 = a_2\rho_2 \circ a_1\rho_1$$

Simple Triptychs/Irreducible Representations

Definition

A triptych is *visible* if $M_i \neq 0$ and $M_1 = M_2 A_2$, $M_0 = M_2(A_2 * A_1)$.

Theorem (W.)

The triptych is visible simple if, and only if, every nonzero is a **unit***:*

$$(\forall m_2) \qquad m_2 \neq 0 \Rightarrow (m_2 A_2) A_1 = M_2(A_2 * A_1).$$

Properties of the representations

Further properties

- Nakayama's lemma.
- Shur's lemma.
- Induction and restriction.
- Morita condensation.

Open problems

Develop characters, blocks, and reciprocity theorems.

We use these to seed filter refinements!

Satisfaction

Satisfaction

$|t\rangle : V_1 \times \cdots \times V_\daleth \rightarrowtail V_0$ multilinear.
$p = \sum_e \lambda_e x_1^{e_1} \cdots x_\daleth^{e_\daleth} x_0^{e_0}$ polynomial.
$\omega = (\omega_1, \ldots, \omega_\daleth, \omega_0) \in \prod_a \mathrm{End}(V_a)$ operator.

Def.

$|t\rangle$ satisfies p at ω if for every $\langle v| = \langle v_1| \cdots \langle v_\daleth|$

$$0 = \langle v|\, p(\omega)\, |t\rangle = \sum_{\lambda_e} \lambda_e \langle v_1 \omega_1^{e_1}, \ldots, v_n \omega_\daleth^{e_\daleth} |t\rangle \omega_0^{e_0}.$$

Examples of satisfaction

Identity	Polynoimal	Operator
$(u\lambda)f = (uf)\lambda$	$x_1 - x_0$	Linear

Pf. Put $\langle u|t\rangle := uf$, $p = x_1 - x_0$.

$$\begin{aligned} 0 &= (u\lambda)f - (uf)\lambda \\ &= \langle u\lambda|t\rangle - \langle u|t\rangle\lambda \\ &= \langle u|p(\lambda,\lambda)|t\rangle. \end{aligned}$$ □

Examples of satisfaction

Identity	Polynoimal	Operator
$(u\lambda)f = (uf)\lambda$	$x_1 - x_0$	Linear
$\langle uX \vert v\rangle = \langle u \vert vX^*\rangle$	**?**	Adjoint

Examples of satisfaction

Identity	Polynoimal	Operator
$(u\lambda)f = (uf)\lambda$	$x_1 - x_0$	Linear
$\langle uX\|v\rangle = \langle u\|vX^*\rangle$	$x_1 - x_2$	Adjoint
$\langle uX\|v\rangle = \langle u\|X^*v\rangle$	$x_1 - \bar{x}_2$	

Pf. Put $\langle u, v|t\rangle := \langle u, v\rangle$, $p = x_1 - x_2$.
$0 = \langle u, v|p(X, X^*)|t\rangle = \langle uX, v\rangle - \langle u, vX^*\rangle$ □
Convenience use $\bar{x}_a$ to denote left action.

Examples of satisfaction

Identity	Polynoimal	Operator
$(u\lambda)f = (uf)\lambda$	$x_1 - x_0$	Linear
$\langle uX \mid v\rangle = \langle u \mid vX^*\rangle$	$x_1 - x_2$	Adjoint
$\langle uX \mid v\rangle = \langle u \mid X^*v\rangle$	$x_1 - \bar{x}_2$	
$\langle \lambda u \mid v\rangle = \lambda\langle u \mid v\rangle = \langle u \mid \lambda v\rangle$	$\{\bar{x}_1 - \bar{x}_0, \bar{x}_2 - \bar{x}_0\}$	Bilinear
?	$x_1 + x_2 - x_0$	**?**

Examples of satisfaction

Identity	Polynoimal	Operator
$(u\lambda)f = (uf)\lambda$	$x_1 - x_0$	Linear
$\langle uX\|v\rangle = \langle u\|vX^*\rangle$	$x_1 - x_2$	Adjoint
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$\langle \lambda u\|v\rangle = \lambda\langle u\|v\rangle = \langle u\|\lambda v\rangle$	$\{\bar{x}_1 - \bar{x}_0, \bar{x}_2 - \bar{x}_0\}$	Bilinear
$(u \cdot v)\delta = u\delta \cdot v + v \cdot v\delta$	$x_1 + x_2 - x_0$	Derivation

Examples of satisfaction

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$\langle \lambda u \vert v\rangle = \lambda\langle u \vert v\rangle = \langle u \vert \lambda v\rangle$	$\{\bar{x}_1 - \bar{x}_0, \bar{x}_2 - \bar{x}_0\}$	Bilinear
$(u \cdot v)\delta = u\delta \cdot v + v \cdot v\delta$	$x_1 + x_2 - x_0$	Derivation
$\langle uX \vert vX\rangle = \langle u \vert v\rangle$	$\{x_1 x_2 - 1, x_0 - 1\}$	Isometry
$\omega(u * v) = \omega'(u) * \omega''(v)$	$\bar{x}_1\bar{x}_2 - \bar{x}_0$	Homotopism

$S \subset T$, $P \subset K[X]$, $\Delta \subset \prod_a \mathrm{End}(V_a)$.

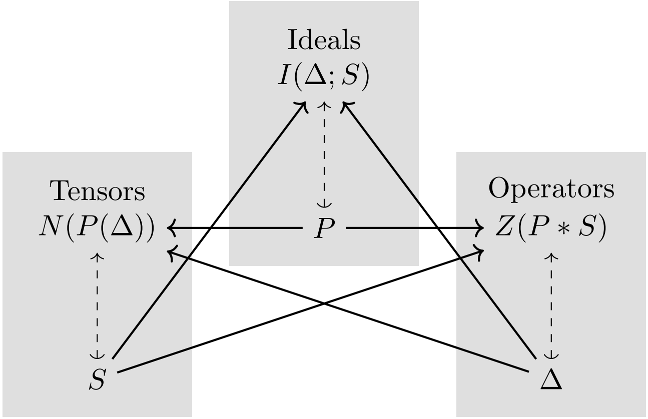
$$N(P(\Delta)) = \{t : P(\Delta)\,|t\rangle = 0\}$$
$$I(\Delta; S) = \{p : p(\Delta)\,|S\rangle = 0\}$$
$$Z(P * S) = \{\omega : p(\omega)\,|S\rangle = 0\}.$$

Correspondence Theorem. First-Maglione-W.

$N(P(\Delta))$ is a subspace, $I(\Delta; S)$ is an ideal, $Z(P * S)$ is an affine-zero set. They satisfy:

$$S \subset N(P(\Delta)) \quad \Leftrightarrow \quad P \subset I(\Delta; S) \quad \Leftrightarrow \quad \Delta \subset Z(P * S).$$

Tensor-Ideal-Operator correspondence



Immediate consequences of tensor theory

Densors

Derivations $\mathrm{Der}(S)$ and **densors** $⦇ S ⦈$ are:

$$\mathrm{Der}(S) = \bigcap_{s \in S} \left\{ \delta : \langle v|s\rangle\delta = \sum_a \langle v_{\widehat{a}}, v_a\delta|s\rangle \right\}.$$

$$⦇ S ⦈ = \{t : \mathrm{Der}(S) \subset \mathrm{Der}(t)\}.$$

Densors are the universal linear tensor space (FMW)

Let $|K| > n$. If $P = (p_1, \ldots, p_m)$, $p_i = \sum_a \lambda_{ia} x_a$, & $\forall a \exists i$, $\lambda_{ai} \neq 0$, then

$$Z(P * S) \hookrightarrow \mathrm{Der}(S) \qquad ⦇ S ⦈ \hookrightarrow N(P(Z(P * S))).$$

Weakly-associative product on $\mathrm{End}(V)$ means $\exists(s,t) \in \mathbb{P}^1(K)$:

$$\omega \bullet \tau = s\omega\tau + t\tau\omega.$$

All linear tensor spaces are over Lie algebras (FMW)

If $p = \lambda_0 x_0 + \cdots + \lambda_n x_n$ then

1. $Z(p * t) \hookrightarrow \prod_a \mathfrak{g}l(V_a)$ as a Lie subalgebra.
2. If $Z(p * t)$ admits a weakly-associative product in every component then all but at most 2 components are Lie.
3. $Z(p * t)$ admits an associative product if, and only if, $n \leq 1$.

Low rank densors are the things we call "simple"

Tensor	Dim. Tensor Space	Dim. Densor
abc-Matrix multiplication	$a^2b^2c^2$	1
Azumaya algebras	$\dim^3 A$	1
Irred. $\mathfrak{sl}_2$-modules	$3d^2$	1
Irred. A_n-modules	$O(n^2d^2)$	1
Irred. B_n-modules	$O(n^2d^2)$	1
Irred. G_2-modules	$14d^2$	1
Octonions	512	1
Albert Algebras	19683	5

And many more collapse as well.

Singularities

All across finite and infinite geometry products without singularities are the building blocks. They are hard to find.

Thm. (FMW)

Fix an infinite field. For every point $\langle U|$ in the product of Grassmannians $\prod_a G(V_a, k_a)$, let

$$\varpi(\langle U|) = \{\pi : \pi^2 = \pi, \mathrm{im}\ \pi = \langle U|\}.$$

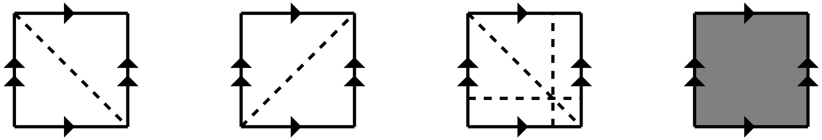
Then $I(\varpi(\langle U|); t)$ is a radical monomial ideal. Furthermore $I(\varpi(\langle U|); t) = (0)$ if, and only if, $\langle v|t\rangle \neq 0$.

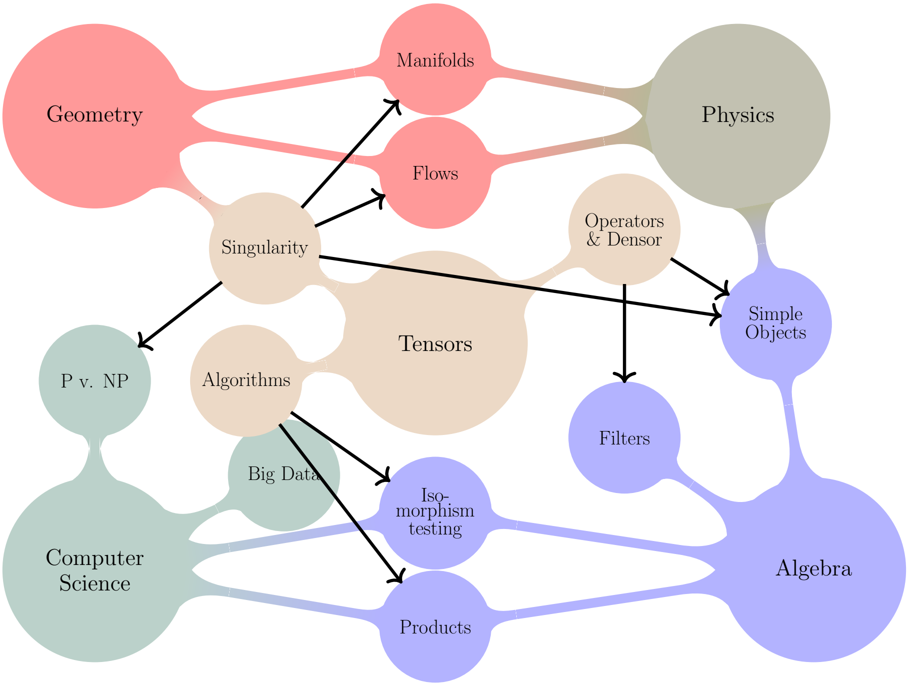
Singularities have structure!

Singularity manifolds for $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrowtail \mathbb{R}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} (0) & (x_1 x_2) \\ (x_1 x_2) & (0) \end{bmatrix} \begin{bmatrix} (x_1 x_2) & (0) \\ (0) & (x_1 x_2) \end{bmatrix} \begin{bmatrix} (0) & (x_2) \\ (x_1) & (x_1, x_2) \end{bmatrix} \begin{bmatrix} (x_1, x_2) & (x_1, x_2) \\ (x_1, x_2) & (x_1, x_2) \end{bmatrix}$$





Geometry
Manifolds
Physics
Flows
Singularity
Operators & Densor
Tensors
Simple Objects
P v. NP
Algorithms
Filters
Big Data
Iso-morphism testing
Computer Science
Algebra
Products

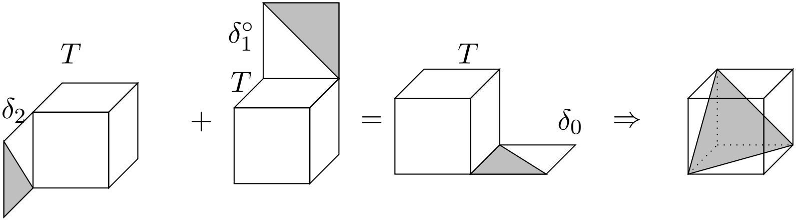
Summary

- Mathematicians, Computer Scientist, and Data Sciences are **struggling** to understand tensors.
- New Perspective:
  - Tensors: a 2-category where nearly all non-associative techniques apply.
  - Tensor analysis, algebraic geometry, and operator theory are in correspondence.
- Current Applications
  - Tensors products are universal over Lie algebras.
  - Simple non-associative constructions are small rank densors.
  - Singularity manifolds now explore tensors as geometries.

Open Problems

1. Find a quadratic variation for characteristic 2.
2. Classify rank 1 densor spaces.
3. Develop characters, blocks, and reciprocity theorems.
4. Better understanding of nonsingular tensors.

The affect of singular operators on a the shape of a tensor.



$\Leftarrow$