Math 369, Sample Final Exam

Note: this is just a sample and should be handled with care. In the final you might expect to be asked to solve other type of problems, too, covering for instance the singular value decomposition, least square problems, row/column/null space, orthogonal subspaces, hermitian matrices and so on. The review sheets for midterms and the review lectures of the last week of classes are a good guide for the preparation of the final exam.

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1. (a) Consider the two vectors $a = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$. Compute the angle between a and b.

   (b) Show that the following two vectors are orthogonal.

$$q_1 = \begin{bmatrix} .8 \\ .6 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} -.6 \\ .8 \\ 0 \end{bmatrix}.$$

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2. Prove that $S = \left\{ v \in \Re^2 : v = \begin{bmatrix} x \\ 2x \end{bmatrix} \right\}$ is a subspace of $\Re^2$.

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3. Determine whether or not the following vectors are linearly independent in $\Re^3$ and check whether or not they form a basis in $\Re^3$. Explain why.

   a) $\vec{v_1} = (1, 0, 0)$, $\vec{v_2} = (0, 1, 0)$, $\vec{v_3} = (1, 0, 1)$, $\vec{v_4} = (1, 2, 3)$;

   b) $\vec{v_1} = (2, 1, -2)$, $\vec{v_2} = (3, 2, -2)$, $\vec{v_3} = (2, 2, 0)$.

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4. Let $\mathbf{u_1}, \mathbf{u_2}$ be an orthonormal basis for C^2 and let $\mathbf{z} = (4 + 21)\mathbf{u_1} + (6 - 5i)\mathbf{u_2}$

   (a) What are the values of $\mathbf{u_1^H z}, \mathbf{z^H u_1}, \mathbf{u_2^H z}, \mathbf{z^H u_2}$?

   (b) determine the value of $||z||$

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5. Decide if each $h : \Re^3 \longrightarrow \Re^2$ below is a linear transformation. If yes, then prove it and if it isn't then state a condition that it fails to satisfy.

   a)

$$h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 2x + y \\ 3y - 4z \end{pmatrix};$$

   b)

$$h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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6. Find an orthonormal basis for this subspace of $\Re^4$

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ y \end{pmatrix} \in \Re^4 \mid x - y - z + w = 0,\ x + z = 0 \right\}$$

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7. Given $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ a & b & c \end{bmatrix}$ find $det(\lambda I - A)$. Pick a, b and c such that

$$det(\lambda I - A) = \lambda^3 - 21\lambda - 6.$$

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8. Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be the linear transformation defined as $T(x, y) = (x + 2y, 3x + 2y)^T$. Let E be the standard basis $\{(1, 0)^T, (0, 1)^T\}$, B the basis $\{\mathbf{u_1}, \mathbf{u_2}\} = \{(1, 3)^T, (2, 5)^T\}$

   (a) Write down the matrix A representing T with respect to the standard basis E for $\mathbf{R}^2$

   (b) Write down the matrix H representing T with respect to the basis B

   (c) Using the matrix H find the coordinates of $T(\mathbf{u_1} + \mathbf{u_2})$ with respect to the basis B

   (d) Find the change of basis matrices A_1 from basis E to basis B, and A_2 from basis B to basis E and verify that $H = A_1 A A_2$

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9. (a) Given $A = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$, find $det(\lambda I - A)$.

   (b) For each eigenvalue, find an eigenvector whose components are all integers. With these eigenvectors, define a matrix S.

   (c) Find the inverse of S.

   (d) Diagonalize A with a similarity transformation.

   (e) Using the results in (d), calculate A^4.