FMI
10

1.2.3 The Material Derivative

Characteristics of the motion (as density, velocity, pressure etc), denoted here generically by F, can be represented using both Eulerian and Lagrangian descriptions,

$$F^E(\mathbf{x}, t) = F^L(\mathbf{x_0}, t). \quad (1.13)$$

The rate of change of F, as attached to the particle $\mathbf{x_0}$, is

$$\frac{dF^L}{dt} = \frac{dF^L}{dt}(\mathbf{x_0}, t). \quad (1.14)$$

The rate of change of F, as seen by an observer sitting in $P(\mathbf{x})$ is

$$\frac{dF^E}{dt} = \frac{dF^E}{dt}(\mathbf{x}, t). \quad (1.15)$$

But

$$F^L(\mathbf{x_0}, t) = F^E(\mathbf{x}^L(\mathbf{x_0}, t), t) \quad (1.16)$$

and, using the chain rule,

$$\frac{dF^L}{dt} = \frac{\partial F^E}{\partial t} + \frac{\partial F^E}{\partial x_1^L}\frac{dx_1^L}{dt} + \frac{\partial F^E}{\partial x_2^L}\frac{dx_2^L}{dt} + \frac{\partial F^E}{\partial x_3^L}\frac{dx_3^L}{dt}. \quad (1.17)$$

We observe that

$$\frac{dx_i^L}{dt} = v_i, \; i = 1 \ldots 3, \quad (1.18)$$

where v_i, $i = 1 \ldots 3$, are the coordinates of the velocity $\mathbf{v}$, therefore we can write (1.17) as

$$\frac{dF^L}{dt} = \frac{\partial F^E}{\partial t} + v_1\frac{\partial F^E}{\partial x_1} + v_2\frac{\partial F^E}{\partial x_2} + v_3\frac{\partial F^E}{\partial x_3}, \quad (1.19)$$

or, equivalently,

$$\frac{dF^L}{dt} = \frac{\partial F^E}{\partial t} + (\mathbf{v} \cdot \nabla)F^E, \quad (1.20)$$

where

$$\nabla = \frac{\partial}{\partial x_1}\mathbf{i_1} + \frac{\partial}{\partial x_2}\mathbf{i_2} + \frac{\partial}{\partial x_3}\mathbf{i_3}. \quad (1.21)$$

We finally obtain the **material derivative** formula

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla)F. \quad (1.22)$$

Velocity and Acceleration

in Lagrangian coordinates: $\quad v_i = \frac{dx_i}{dt} = \frac{\partial x_i}{\partial t}$

$x_i = x_i(\vec{x}_0, t),\ i=1,2,3 \qquad a_i = \frac{d^2 x_i}{dt^2} = \frac{\partial^2 x_i}{\partial t^2}$,

In Eulerian coordinates:

$$v_i = V_i(\vec{x}, t)$$

$$a_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + (\vec{v}\cdot\nabla)\, v_i \Rightarrow \boxed{\vec{a} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}} \qquad (1\text{-}28)$$

Example: Consider the 1-D flow in Example 2.

$$a^E = \frac{dv^E}{dt} = \frac{\partial v^E}{\partial t} + v^E \frac{\partial v^E}{\partial x}, \quad \text{so}$$

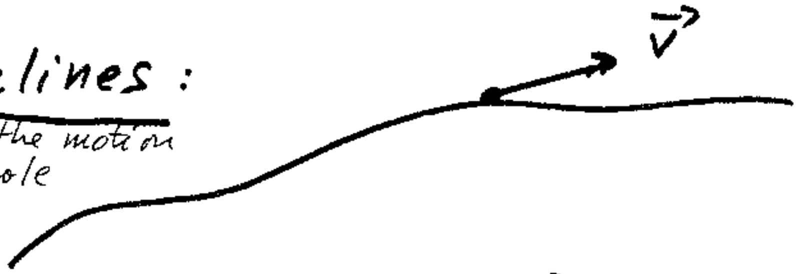
$$a^E = \frac{Dv^E}{Dt} = \left[\frac{2x}{1+t^2} - \frac{4xt^2}{(1+t^2)^2}\right] + \frac{2xt}{1+t^2}\cdot\frac{2t}{1+t^2} = \frac{2x}{1+t^2}$$

as before!

Applications: See HW 1, Part 1

Streamlines:

map of the motion as a whole



$\vec{r} = \vec{r}(s)$

at a constant time t, $\vec{v}$ tangent to the curve $\Rightarrow$

$$\frac{d\vec{r}}{ds} = \lambda \vec{V}(\vec{r}, t)\Big|_{t=\text{const}}.$$

Example: $\vec{V} = v_0 \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix}$; $\begin{cases} \frac{dx}{ds} = v_0 \\ \frac{dy}{ds} = t v_0 \\ \frac{dz}{ds} = 0 \end{cases} \Rightarrow \begin{cases} y = tx + k,\ k = \text{const} \\ dz = 0 \end{cases}$ lines.

1.2.4. Euler's Theorem

Euler's theorem gives the material derivative of J as

$$\boxed{\mathring{J} = J \operatorname{div} \vec{v}} \qquad (1.24)$$

Proof. Since J is a determinant, we have

$$\frac{dJ}{dt} = \sum_{i=1}^{3} J_i$$

where each of J_i, $i=1,2,3$, is a determinant obtained from J by differentiating the 1st, 2nd and 3rd row (column) respectively. Therefore

$$J_1 = \begin{vmatrix} \frac{d}{dt}\left(\frac{\partial x_1}{\partial x_1^\circ}\right) & \frac{\partial x_2}{\partial x_1^\circ} & \frac{\partial x_3}{\partial x_1^\circ} \\ \frac{d}{dt}\left(\frac{\partial x_1}{\partial x_2^\circ}\right) & \frac{\partial x_2}{\partial x_2^\circ} & \frac{\partial x_3}{\partial x_2^\circ} \\ \frac{d}{dt}\left(\frac{\partial x_1}{\partial x_3^\circ}\right) & \frac{\partial x_2}{\partial x_3^\circ} & \frac{\partial x_3}{\partial x_3^\circ} \end{vmatrix}$$

and we have similar formulas for J_1 and J_2. But

$$\frac{d}{dt}\left(\frac{\partial x_1}{\partial x_1^\circ}\right) = \frac{\partial}{\partial x_1^\circ}\left(\frac{dx_1}{dt}\right) = \frac{\partial v_1}{\partial x_1^\circ} = \frac{\partial v_1}{\partial x_1}\underbrace{\frac{dx_1}{dx_1^\circ}}_{=\frac{\partial x_1}{\partial x_1^\circ}\ \text{etc}} + \frac{\partial v_1}{\partial x_2}\frac{\partial x_2}{\partial x_2^\circ} + \frac{\partial v_1}{\partial x_3}\frac{\partial x_3}{\partial x_3^\circ}$$

therefore

$$J_1 = \begin{vmatrix} \sum \frac{\partial v_1}{\partial x_i}\frac{\partial x_i}{\partial x_1} & \frac{\partial x_2}{\partial x_1^\circ} & \frac{\partial x_3}{\partial x_1^\circ} \\ \sum \frac{\partial v_1}{\partial x_i}\frac{\partial x_i}{\partial x_2} & \frac{\partial x_2}{\partial x_2^\circ} & \frac{\partial x_3}{\partial x_2^\circ} \\ \sum \frac{\partial v_1}{\partial x_3}\frac{\partial x_3}{\partial t} & \frac{\partial x_2}{\partial x_3^\circ} & \frac{\partial x_3}{\partial x_3^\circ} \end{vmatrix} = \sum_{i=1}^{3} \frac{\partial v_1}{\partial x_i} \begin{vmatrix} \frac{\partial x_i}{\partial x_1^\circ} & \cdots \\ \vdots & \end{vmatrix} \rightarrow$$

so, finally

$$\mathring{J} = \frac{\partial v_1}{\partial x_1} J + \frac{\partial v_2}{\partial x_2} J + \frac{\partial v_3}{\partial x_3} J,$$

as only one determinant in J_1 is nonzero (for $i=1$), and similarly, for J_2, J_3. □

1.25 The Material Derivative of an Integral over a Material Domain

We want to calculate

$$\frac{d}{dt}\int_D F(\vec{x},t)\,dv, \qquad (1.25)$$

where $D\subset\mathbb{R}^3$ is a material domain and $F\in C^1(D\times T)$.
We use the diffeomorphism $\vec{x}=\mathcal{X}(\vec{x}_0,t)$ to integrate on D_0, where $D_0=\mathcal{X}^{-1}(D)$, with the help of Liouville's theorem

$$\int_D F(\vec{x},t)\,dV = \int_{D_0} F(\vec{x}(\vec{x}_0,t),t)\,J\,dV_0 ,$$

therefore

$$\frac{d}{dt}\left(\int_D F\,dV\right) = \int_{D_0}\frac{d}{dt}(FJ)\,dV_0 =$$

$$= \int_{D_0}\left(\frac{dF}{dt}J + F\frac{dJ}{dt}\right)dV_0 = \int_{D_0}\left(\frac{dF}{dt}J + FJ\,\mathrm{div}\,\vec{v}\right)dv_0 =$$

$$= \int_{D_0}\left(\frac{dF}{dt} + F\,\mathrm{div}\,\vec{v}\right)J\,dv_0 = \int_D\left(\frac{dF}{dt} + F\,\mathrm{div}\,\vec{v}\right)dv$$

so we obtain the formula

$$\frac{d}{dt}\int_D F\,dV = \int_D\left(\frac{dF}{dt} + F\,\mathrm{div}\,\vec{v}\right)dv , \qquad 1.26$$

or, (equivalently) using the material derivative formula for $\frac{dF}{dt}$,

$$\frac{d}{dt}\int_D F\,dv = \int_D\left[\frac{\partial F}{\partial t} + \mathrm{div}(F\vec{v})\right]dV \qquad 1.27$$

Remark 1. Here we used $\boxed{\mathrm{div}\,(F\vec{v}) = (\vec{v}\cdot\nabla)F + F\,\mathrm{div}\,\vec{v}}$ (1.28)

proof: $\mathrm{div}(F\vec{v}) = \sum v_i\frac{\partial F}{\partial x_i} + \sum\frac{\partial v_i}{\partial x_i}F$

Remark 2: (1.26) is also called Reynolds Formula for continuous media.

Remark 3: Reynolds formula for shock waves - to be discussed later.

FMI/14

1.3 Cauchy's Principle. Stress tensor.

Cauchy enunciated the principle that, within a body, the forces that an enclosed volume imposes on the remainder of the material must be in equilibrium with the forces upon it from the remainder of the body.

Consider the subsystem $\mathcal{P}$ of the material system $\mathcal{M}$, and denote by D the support of $\mathcal{P}$ (see Figure 2). As a part of $\mathcal{M}$, $\mathcal{P}$ will *interact* with $\mathcal{M} - \mathcal{P}$. In mechanics the *action* is modelled by *force*. The action of $\mathcal{M} - \mathcal{P}$ on $\mathcal{P}$ is a contact action on Σ, the separation surface between $\mathcal{M} - \mathcal{P}$ and $\mathcal{P}$.

Cauchy's Principle. *There exists a distribution* $\mathbf{t} = \mathbf{t}(\mathbf{x},t)$ *of forces on the surface* Σ, *absolutely continuous of the area, such that the action of* $\mathbf{t}$ *on* $\mathcal{P}$ *is equivalent to the action of* $\mathcal{M} - \mathcal{P}$ *on* $\mathcal{P}$

We call $\mathbf{t}$ the *force density*. The total contact action $\boldsymbol{R}^c$ and momentum $\boldsymbol{M}^c$ are

$$\boldsymbol{R}^c(\mathcal{P}) = \int_D \boldsymbol{t}\, da, \quad \boldsymbol{M}^c(\mathcal{P}) = \int_D \boldsymbol{x} \times \boldsymbol{t}\, da.$$

Cauchy's Postulate. $\mathbf{t}$ *is the same for all surfaces having the same orientation and tangent plane at* $\mathbf{x}$.

We therefore can write

$$\mathbf{t} = \mathbf{t}(\mathbf{n}, \mathbf{x}, t),$$

where $\mathbf{n}$ is the outward normal to Σ at $P(\mathbf{x})$. $\mathbf{t}$ is also called the *stress vector* at time t, point $P(\mathbf{x})$ and corresponding to the surface element with outward normal $\mathbf{n}$.

Remark: The total action of the external forces can be written as

$$\boldsymbol{R}^d(\mathcal{P}) = \int_D \rho \boldsymbol{f}\, dV,$$

where the superscript d stands for distance, $\mathbf{f}$ is the external force density (force per mass unit); ρf is the force per volume unit, as $dm = \rho dV$. Generically,

$$\mathbf{f} = \mathbf{f}(\mathbf{x}, t).$$

If $\mathbf{f} = \mathbf{f}(\mathbf{x})$ the forces are called *steady*; if $\mathbf{f} = const.$, they are called *homogeneous*, like $\mathbf{f} = \mathbf{g}$, the gravitational forces.

Cauchy's Lemma. *If* $\mathbf{t}$ *is continuous with respect to* $\mathbf{x}$, *then*

$$\mathbf{t}(\mathbf{n}, \mathbf{x}, t) = -\mathbf{t}(-\mathbf{n}, \mathbf{x}, t).$$

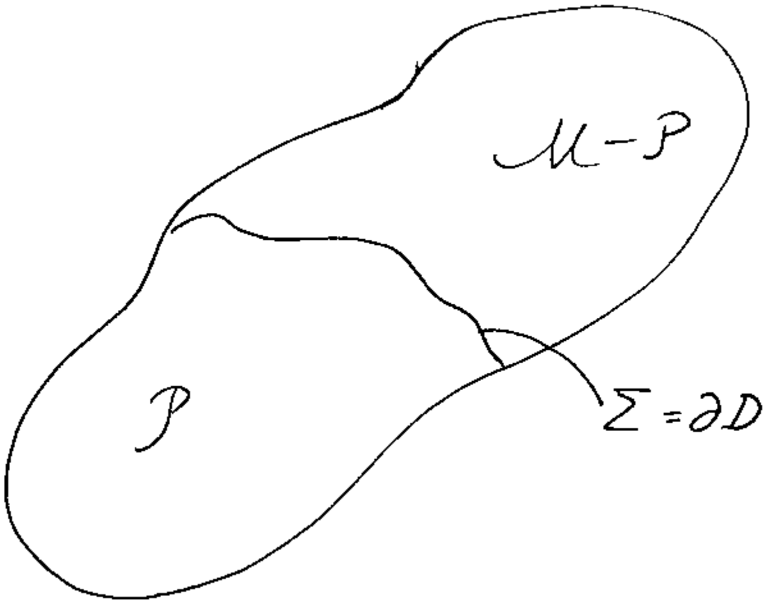


Figure 2.

Remark:

$\vec{t}$ depends on $\vec{n}$ so it cannot represent the stress state at $\vec{x}$. For that, consider $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$ to be the unit vectors along Ox_1, Ox_2 and Ox_3 respectively. Denote by

$$\vec{T}_j(\vec{x}) = \vec{t}(\vec{i}_j, \vec{x}), \quad j = 1,2,3,$$

and by $\quad (T_{ij})_{i=1,2,3}$ their coordinates,

$$\vec{T}_j = (T_{1j}, T_{2j}, T_{3j}), \quad j=1,2,3.$$

Definition: $\quad (T_{ij})_{1\le i,j\le 3}$ is called Cauchy's stress tensor. It characterizes the stress state of the system at $P(\vec{x})$.

2.1. The mass and the density

Let $\mathcal{M}$ be a material system; we associate to it a state quantity $m(\mathcal{M}) > 0$, $m : \mathcal{M} \to R_+$, such that

a) for any division $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ of the system,

$$m(\mathcal{M}) = m(\mathcal{M}_1) + m(\mathcal{M}_2). \tag{2.1}$$

b) the mass of the system, $m(\mathcal{M})$, remains constant during the motion:

$$\frac{dm}{dt} = 0. \tag{2.2}$$

As a consequence,

$$\boxed{m(\mathcal{M}) = \int_{\mathcal{M}} dm} \tag{2.3}$$

We will consider the following hypothesis, for continuous media, namely the mass is a continuous function of $\mathcal{M}$.

As a consequence, one can show, using Radon-Nicodim theorem, which states that any absolutely continuous function on a countable set can be represented as an integral of a density, we have

$$m(\mathcal{P}) = \int_D \rho(\vec{x}, t)\, dV, \tag{2.4}$$

where $\rho(\vec{x}, t)$ is a positive, scalar function, uniquely defined on D, the support of $\mathcal{P}$. $\boxed{\rho(\vec{x}, t)}$ is called the specific mass (the mass of the volume unit), or the density of mass, or, simply, density

Remark 2.1. **The** mass represents a quantitative measure of inertia, or the resistance of a body to a change in motion.

or, equivalently,

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0}\,, \quad (\text{Euler, } 1757) \qquad (2.8)$$

This is the continuity equation in Eulerian description. It expresses the principle of conservation of mass for continuous media / continuous motions.

Remark 2.2

If we multiply (2.7) by J and apply Euler's theorem we obtain

$$J\frac{d\rho}{dt} + \rho J \operatorname{div}\vec{V} = 0 \Rightarrow J\frac{d\rho}{dt} + \rho\frac{dJ}{dt} = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}(\rho J) = 0 \qquad \Rightarrow \rho J = (\rho J)\big|_{t=0} = \rho_0$$

Therefore we get that

$$\boxed{\rho J = \rho_0}\,, \qquad (2.9)$$

namely the continuity equation in Lagrangian description.

Remark 2.3

For incompressible media (flows): the volume remains constant during the motion. Therefore, since the volume of D is

$$V = \int_D dV = \int_{D_0} J\, dV = \int_{D_0} dV \Rightarrow \boxed{J = 1}$$

if $\frac{dV}{dt} = 0$, then $0 = \frac{d}{dt}\int_{D_0} J\, dV = \int_{D_0} \frac{dJ}{dt}\, dV = \int_{D_0} J \operatorname{div}\vec{v}\, dv =$

$= \int_D \operatorname{div}\vec{v}\, dv$, so from the fundamental Lemma we have that $\boxed{\operatorname{div}\vec{v} = 0}$ and $\dot{\rho} = \rho_0$, where ~~the~~ ρ_0 is the initial value. $\dot{\rho} = 0$ means $\rho =$ constant, but the constant can differ from one trajectory to another one.

2.2. Conservation of Mass. Continuity Principle (Continuity Equation).

Common to all continuous media is the principle that matter can be neither created, nor destroyed. In continuum mechanics, this is a nontrivial statement and leads, via (2.2), to the continuity equati[on]

We will need the following lemma:

Fundamental Lemma.

Let $f: \mathscr{D} \to \mathbb{R}$, $\mathscr{D} \subset \mathbb{R}^3$, a continuous function such that

$$\int_D f(\vec{x})\,dV = 0 \tag{2.5}$$

for any subdomain $D \subset \mathscr{D}$. Then $f(\vec{x}) \equiv 0$ in $\mathscr{D}$.

Proof: assume there exists an $\vec{x}_0 \in \mathscr{D}$, s.t. $f(\vec{x}_0) \neq 0$. Assume, for simplicity, that $f(\vec{x}_0) > 0$. Due to the continuity, $f(\vec{x}) > 0$ in a whole neighbourhood D of $\vec{x}_0$, which contradicts (2.5) $\square$.

From (2.2), we have

$$\frac{d}{dt}\int_D \rho(\vec{x},t)\,dV = 0, \quad (\forall)\ D \subset \mathscr{D}. \tag{2.6}$$

and from Reynold's formula

$$\int_D \left(\frac{d\rho}{dt} + \rho \operatorname{div}\vec{v}\right) dV = 0 \quad (\forall)\ D \subset \mathscr{D}.$$

If $\rho(\vec{x},t)$ is $C^1(\mathscr{D} \times T)$, from the fundamental Lemma we have that

$$\boxed{\frac{d\rho}{dt} + \rho \operatorname{div}\vec{v} = 0} \tag{2.7}$$