FMI
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1 General Theory

1.1 Introduction

Fluid mechanics is the subdiscipline of **continuum mechanics** that studies fluids, that is, liquids and gases.

Fluid dynamics is the sub-discipline of **fluid mechanics** dealing with fluids (liquids and gases) in motion. It has several subdisciplines itself, including aerodynamics (the study of gases in motion) and hydrodynamics (the study of liquids in motion). Fluid dynamics has a wide range of applications, including calculating forces and moments on aircraft, determining the mass flow rate of petroleum through pipelines, predicting weather patterns, understanding nebulae in interstellar space and reportedly modelling fission weapon detonation. Some of its principles are even used in traffic engineering, where traffic is treated as a continuous fluid.

The continuum hypothesis. Fluids are composed of molecules that collide with one another and solid objects. The continuum assumption, however, considers fluids to be continuous. That is, properties such as density, pressure, temperature, and velocity are taken to be well-defined at "infinitely" small points, defining a reference element of volume (REV), at the geometric order of the distance between two adjacent molecules of fluid. Properties are assumed to vary continuously from one point to another, and are averaged values in the REV. The fact that the fluid is made up of discrete molecules is ignored.

The continuum hypothesis is basically an approximation; under the right circumstances, the continuum hypothesis produces extremely accurate results. Those problems for which the continuum hypothesis does not allow solutions of desired accuracy are solved using statistical mechanics.

1.2 Kinematics of Fluids

Definition 1. Continuous medium - a material medium filling up completely a region in space.

Let's assume that the physical support is a simply connected domain $\mathcal{D} \in \Re^3$. Every point in $\mathcal{D}$ is a material particle; the term particle here refers to a very small section of the continuum material, like a drop of water, or few grains of sand. In a mathematical model a particle is vanishingly small and identifiable with a (geometrical) point. We will understand by **material particle** a (geometrical) point with a mass associated to it.

Definition 2. Fluid - a continuously deformable medium such that the distance between any two particles can become as larger as possible with time.

Problem: the mathematical characterization of fluid motion.

Tools: deformation tensor, constitutive laws.

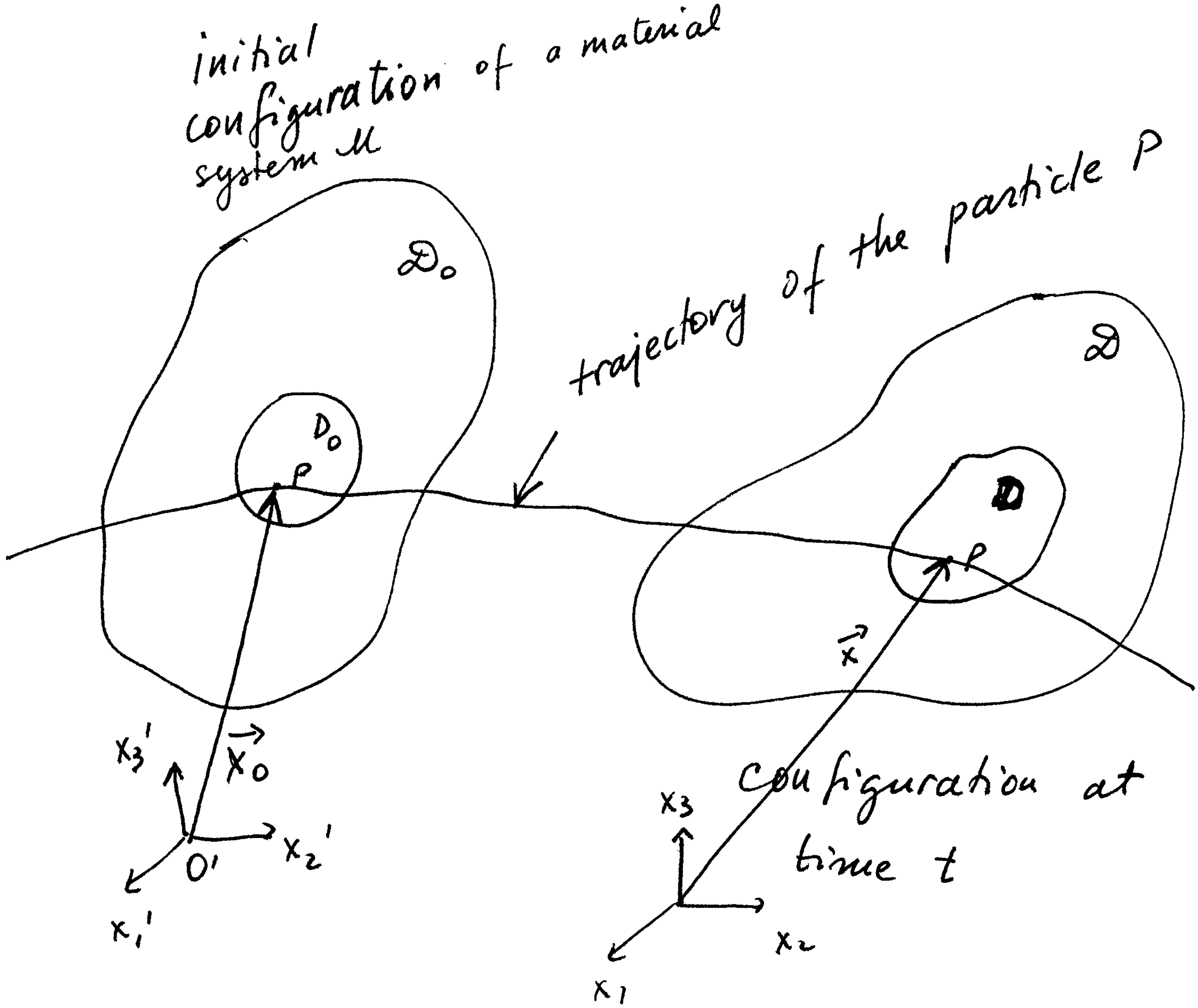

Figure 1.1

$\vec{x}$ = the position vector of the material particle P at time t

$\vec{X}_0$ - the position vector of the particle at $t = t_0$.

We define the motion of a particle P (see Fig. 1.1) with the help of a vector function χ,

$$\boxed{\vec{x} = \chi(\vec{x}_0, t), \quad \chi : \mathcal{D}_0 \times T \to \mathcal{D},} \qquad (1.1)$$

where $T = [t_0, t_1]$ is the time interval we study the motion.
If $\vec{x}_0$ is kept fixed and t allowed to vary, (1.1) represents the law of motion of a particle P having the position vector $\vec{x}_0$ at $t = t_0$, and defines its trajectory (see Fig 1.1).
We define the velocity and acceleration of P as

$$\boxed{\vec{v} = \frac{d\vec{x}}{dt}, \quad \vec{a} = \frac{d\vec{v}}{dt}} \qquad (1.2)$$

Remark 1

The variation of the function χ over $\mathcal{D}_0 \times T$ defines the motion of the system $\mathcal{U}$ during the time interval. The variation of χ with t characterizes the motion; the variation of χ with $\vec{x}_0$ characterizes the deformation

Remark 2

We will consider, in the following, a unique refference system $Ox_1x_2x_3$ (i.e. $O' = O$ in Fig. 1.1). With respect to this system we denote the coordinates of $\vec{x}$ and $\vec{x}_0$ by

$$\boxed{\vec{x} = (x_1, x_2, x_3), \quad \vec{x}_0 = (x_1^\circ, x_2^\circ, x_3^\circ)} \qquad (1.3)$$

We therefore have

$$\boxed{x_i = x_i(x_1^\circ, x_2^\circ, x_3^\circ, t), \; i = 1, 2, 3,} \qquad (1.4)$$

where $x_i : \mathcal{D}_0 \times T \to \mathbb{R}$.

Remark 3

We will consider functions $\mathcal{X}$ of class $C^2(\mathcal{D}_0 \times T)$, i.e. continuous and with 1^{st} and 2^{nd} derivatives continuous. The meaning of this hypothesis: particles close enough at $t = t_0$ will remain neighbors at any time t.

Problem $\boxed{\text{Has the function } \mathcal{X} \text{ an inverse?}}$

Solution: Solve
$$\begin{cases} x_1 = x_1(x_1^\circ, x_2^\circ, x_3^\circ, t) \\ x_2 = x_2(x_1^\circ, x_2^\circ, x_3^\circ, t) \\ x_3 = x_3(x_1^\circ, x_2^\circ, x_3^\circ, t) \end{cases}$$
for $x_1^\circ, x_2^\circ, x_3^\circ$ (*)

Denote by
$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(x_1^\circ, x_2^\circ, x_3^\circ)} \qquad (1.5)$$

the determinant
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial x_1^\circ} & \frac{\partial x_2}{\partial x_1^\circ} & \frac{\partial x_3}{\partial x_1^\circ} \\ \frac{\partial x_1}{\partial x_2^\circ} & \frac{\partial x_2}{\partial x_2^\circ} & \frac{\partial x_3}{\partial x_2^\circ} \\ \frac{\partial x_1}{\partial x_3^\circ} & \frac{\partial x_2}{\partial x_3^\circ} & \frac{\partial x_3}{\partial x_3^\circ} \end{vmatrix} \qquad (1.6)$$

and assume that
$$\boxed{J \neq 0} \qquad (1.7)$$

all over $\mathcal{D}_0 \times T$, excepting some singular points, curves or surfaces. The implicit functions theorem then guaranties the solvability of the system (*) and the local existence of the inverse

$$\boxed{\vec{X}_0 = \vec{X}_0(\vec{x}, t)} \qquad (1.8).$$

where $x_i^0 : \mathcal{D} \times T \to \mathcal{D}_0$, $i = 1, 2, 3$, of class $C^2(\mathcal{D}, T)$.

Remark 4

The hypothesis $\boxed{J \neq 0}$ is called the continuity hypothesis (axiom), or the hypothesis (axiom) of the indestructibility of the matter:

" An initially finite volume element dV never becomes zero during the motion".

or

" the matter can neither be created nor destroyed

and is common to all continuous media.

1.2 Lagrangian and Eulerian description

The motion of the continuous medium, thus of the fluid, can be described either using

$$\boxed{\text{Lagrangian (initial, material) coordinates } X_1^0, X_2^0, X_3^0,}$$

as in elasticity theory, or with

$$\boxed{\text{Eulerian (current, spatial) coordinates } X_1, X_2, X_3,}$$

as in fluid mechanics.

1.2.1. Lagrangian Description

- actually invented by Euler

Example 1.1 : $\boxed{\text{The one-dimensional motion.}}$

$$X = X^L(X_0, t), \quad v = V^L(X_0, t) = \frac{dX^L}{dt}(X_0, t) \text{ - velocity}$$

where the superscript L denotes a lagrangian function (position, velocity, pressure etc).
In particular $X^L(X_0, t)$ gives the current position of the particle situated in $X = X_0$ at $t = t_0$.

Example 1.2

Suppose that the particles of a continuum move according to

$$X = X^L(X_0, t) = X_0 + X_0 t^2. \tag{1.9}$$

Observe that at $t=0$ $X^L(X_0, 0) = X_0$, so the Lagrangian labelling is correct. The path of each particle is a parabola as t varies, as shown in the space-time diagram Fig. 1.2:

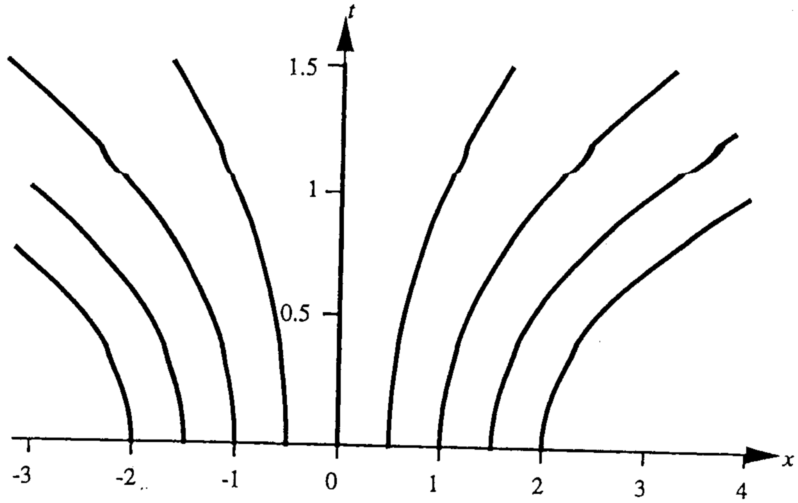

Figure 1.2: Particle paths of the continuum movement used Example 1.2.

Observe that as time progresses the particles spread out. We could imagine this deformation being that of a rod or spring which is being streched. To investiga more aspects of the motion we calculate

$$v^L(X_0,t) = \frac{\partial x^L}{\partial t} = 2X_0 t \quad , \quad a^L(X_0,t) = \frac{\partial v^L}{\partial t} = 2X_0 \ ; \qquad (1.10)$$

it is clear that the particles have constant acceleration, but the acceleration is different for each particle.

1.2.2. Eulerian description

$$\vec{X}_o = \vec{X}_o^{E}(\vec{x},t)\,, \quad \vec{V} = \vec{v}^{E}(\vec{x},t)\,, \quad \rho = \rho^{E}(\vec{x},t)$$

we describe quantities as functions of position $\vec{x}$ and time t

We remark that $\vec{X}_o = \vec{X}_o^{E}(x,t)$ is the inverse of $\vec{x} = \vec{x}^{L}(\vec{X}_o,t)$

Example 1.3

$$x = x^{o} + Ut = x^{L}(X_o,t) \qquad (1.11)$$

represents the translation of a continuum with constant velocity U; in Eulerian description

$$X_o = x_o^{E}(x,t) = x - Ut \qquad (1.12)$$

Example 1.4

Consider further the deformation of Example 1.2.

a) at any time t the following table exists:
(say $t=1$)

particle X_o	location $X = X_o + X_o t^2$	velocity $v = 2X_o t$
-1	-2	-2
0	0	0
1	2	2
2	4	4

Thus at each time, here $t=1$, at each position x, the material has a particular velocity $v^{E}(x,1)$; from the above table we would know that

$$v^{E}(2,1) = 2$$

for example.

b) How do we find $v^E(x,t)$ from a Lagrangian description for a point (x,t) in space time?

Answer: first we invert $X = X_0 + X_0 t^2$ to obtain

$$X_0 = X_0^E(X,t) = \frac{X}{1+t^2}.$$ Secondly, for this

X_0 we must have $v^E(x,t) = v^L(X_0,t)$.

For instance $v^E(4,1) = v^L(2,1) = 4$.

Thus

$$v^E(x,t) = v^L(X^0(x,t),t) = 2\left(\frac{x}{1+t^2}\right)t = \frac{2xt}{1+t^2}$$

c) similarly, for the acceleration,

$$a^E(x,t) = a^L(x_0^E(x,t),t) = 2\cdot\frac{x}{1+t^2} = \frac{2x}{1+t^2}.$$

Remark 5.

Eulerian description: an observer fixed in the position $\vec{x}$

Lagrangian description: an observer attached to a moving particle $\vec{X_0}$.