M560 fall 2007, Assignment 2

Due Friday, September 28

Problem 1: Show that the mappings described below are linear:
(a) $T: \mathbb{C} \to \mathbb{C}$ (with $\mathbb{C}$ regarded as a vector space over $\mathbb{R}$) mapping a complex number into its conjugate

(b) $T: P_5 \to P_8$ defined as $(Tp)(t) = p(t+1) - p(t) + \int_{-1}^{t} s^2 p(s)ds$

Problem 2: *Investigate the validity of the following statement and prove it if it is true, give a counterexample if it is false*: If l is a non-zero scalar linear function on a (not necessarily finite-dimensional) linear space X, and if α is an arbitrary scalar, does there necessarily exist a vector $x \in X$ such that $l(x) = \alpha$?

Problem 3: Show that if $\dim X = 1$ and $T \in \mathscr{L}(X, X)$ then there is $k \in K$ such that $Tx = kx$ for all $x \in X$.

Problem 4: Suppose that U and V are finite-dimensional linear spaces and $S \in \mathscr{L}(V, W)$, $T \in \mathscr{L}(U, V)$. Show that $\dim N_{ST} \le \dim N_S + \dim N_T$.

Problem 5: Let $T: \mathbb{C}^3 \to \mathbb{C}^3$ be defined as

$$T((a_1, a_2, a_3)) = (a_1 - a_2 + ia_3, 2a_1 + ia_2, (2+i)a_1 - a_3)$$

(a) Verify that T is a linear map
(b) Find R_T and N_T (by giving bases for both).

Problem 6: Show that if X is a finite-dimensional space then the space $L(X, X)$ of all linear maps of X into X is finite-dimensional. Find the dimension of $L(X, X)$.

Problem 7: Let $T: P_n \to P_n$ be the linear map such that $Tp(t) = p(t+1)$. Show that if D is the differentiation operator then

$$T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \ldots + \frac{D^{n-1}}{(n-1)!}$$

Problem 8: If A is a linear map on an n-dimensional linear space, then there exists a non-zero polynomial p of degree $\le n^2$ such that $p(A) = 0$.

Problem 9: Let θ be a real number. Show that the following two matrices are similar over the field of complex numbers:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

Problem 10: Let T be a linear operator on $\mathbb{R}^2$ defined by $T(a_1, a_2) = (-a_2, a_1)$. Prove that for every real number c the operator $(T - cI)$ is invertible (without the use of determinants or eigenvalues).