M531 Take Home Exam
Fall, 2002 - Instructor: Dr. Iuliana Oprea
NAME:

This is a take-home exam, and you must work on it alone[1]. You must turn it at the Friday 6 December lecture - no late exams will be accepted. Show your work to get points. You may use software to check your results, but do not use them as substitutes for showing your work; if you use software for computing integrals, you must enclose the full worksheet (code plus results). Attempt a clear presentation of your work, including verbal explanations. Grading will be also based on orderly and transparent presentation.

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1. Consider the following PDEs:

(a)

$$\frac{\partial u}{\partial t} = K(x)\frac{\partial^2 u}{\partial t^2} + q(x),\ 0 < x < L$$

(b)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}[K(x,u)\frac{\partial u}{\partial x}] + \varrho(x)\omega^2(x)u,\ 0 < x < L$$

(c)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nabla^2 u,\ 0 < x < L$$

For each of the above PDEs state which of the following statements apply:

(i) PDE is homogeneous

(ii) PDE is linear

(iii) PDE has variable coefficients

(iv) PDE is nonlinear

2. Find the separated solutions of the wave equation

$$u_{tt} - c^2 u_{xx} = 0,\ \ 0 < x < L,$$

where c is a real constant, that satisfy the following boundary conditions

$$\frac{\partial u}{\partial t}(0,t) = 0,\ \frac{\partial u}{\partial t}(L,t) = 0.$$

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[1]Send an email at juliana@math.colostate.edu if you have questions about it.

3. Consider the following Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0,\ 0 < x < 1$$

$$BC: \ \ X(0) + 2X'(0) = 0,\ X(1) + 2X'(1) = 0$$

(a) Find all eigenvalues and eigenfunctions $\Phi_n(x)$.

(b) As eigenfunctions of a regular Sturm-Liouville problem, $\{\Phi_n\}$ form an *orthogonal* set of functions (see Theorem about properties of Sturm-Liouville problems). Find the corresponding *orthonormal* set of functions $\{\Psi_n\}$. Find the projection of the function $f(x) = 1$ on this orthonormal set $\{\Psi_n\}$. That is, find the coefficients c_n such that

$$f(x) = 1 = \sum_n c_n \Psi_n(x).$$

Simplify your result as much as possible.

4. Consider the heat equation $u_t = u_{xx}$, where $0 < x < 1$, $t > 0$, with the boundary conditions[2]

$$u(0,t) + 2u_x(0,t) = 0$$

$$u(1,t) + 2u_x(1,t) = 0$$

and the initial condition

$$u(x,0) = f(x), 0 < x < 1.$$

(a) Find the solution of the above problem for general continuous $f(x)$, using separation of variables.

(b) What is the solution in the special case when $f(x) = 1$? Give full details.

Xcredit: describe the behavior of the solution as $t \longrightarrow \infty$.

5. Solve the initial-value problem $u_t = Ku_{xx}$ for $t > 0$, $0 < x < L$, with the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$, $t > 0$, and the initial condition $u(x,0) = 0$, $0 < x < L$.

6. Xcredit Consider the heat conduction problem

$$\frac{\partial}{\partial x}(\kappa(x)\frac{\partial T(x,t)}{\partial x}) = \frac{\partial T}{\partial t}.$$

The ends $x = 0$ and $x = L$ are kept at the temperatures $T = 0$ and $T = T_0$, respectively, and the thermal conductivity depends on the position

$$\kappa(x) = \frac{\kappa_0}{1 + \alpha x},\ \alpha > 0,\ \kappa_0 > 0.$$

Find the steady state temperature T_S.

[2]The boundary condition at $x = 1$ is in the form of Newton's law of cooling; the rate that heat escapes from the rod at this end is proportional to the temperature at the end of the rod. On the other hand, the boundary condition at $x = 0$ is "backwards": the rate the heat enters the rod at this end is proportional to the temperature at the end of the rod.