M340: Solutions; Sample Exam (2) 2006.

(1.) $x^2 y'' - x(x+2)y' + (x+2)y = 2x^3$

a) $\{x, xe^x\}$ FSS

b) y_p – using variation of parameters:

(#4) $\boxed{y'' - \frac{x+2}{x} y' + \frac{x+2}{x^2} y = 2x}$ $\quad p(x) = -\frac{x+2}{x}$, $q(x) = \frac{x+2}{x^2}$, $g(x) = 2x$

$y_1(x) = x$, $y_2(x) = xe^x$

a) x, xe^x solutions: (5 pts)

$W\{x, xe^x\} \neq 0$: (5 pts) $\quad W\{x, xe^x\} = x^2 e^x$

b) (5 pts)

$$\begin{cases} u_1(x) = \int \frac{-g(x) y_2(x)}{W(y_1, y_2)} dx = \int \frac{(-2x)xe^x}{x^2 e^x} = -2x \\ u_2(x) = \int \frac{(2x)x}{x^2 e^x} dx = -2e^{-x} \end{cases}$$

$$\boxed{y(t) = c_1 x + c_2 xe^x + x u_1(x) + xe^x u_2(x)}$$ 5 pts.

(#)2: a) $y'' - y = e^{2x} + xe^x$

$r^2 - 1 = 0$ $\quad r_1 = 1, r_2 = -1$ $\Rightarrow$ $y_G = c_1 e^x + c_2 e^{-x}$

$\{FSS\}: \{e^{x}, e^{-x}\}$ $\quad y_p = Ae^{2x} + Bxe^x$

b) $y'' + y = x^2 - \cos x$

$r^2 + 1 = 0$ $\quad \{\cos x, \sin x\}$

$$y_p = (Ax^2 + Bx + C) + x\{A\cos x + B\sin x\}.$$

(# 3) a) $[s^2 Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 3Y(s) = \frac{1}{s^2}$

$(s^2 - 4s + 3)Y(s) = \frac{1}{s^2} + 2$

$$Y(s) = \frac{1}{s^2(s-1)(s-3)} + \frac{2}{(s-1)(s-3)}$$

We have:

$$\frac{2}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3}$$

$2 = A(s-3) + B(s-1)$

$s = 3 \Rightarrow 2 = 2B \Rightarrow B = 1$

$s = 1 \Rightarrow 2 = (-2)A \Rightarrow A = -1$

Therefore $\boxed{\dfrac{2}{(s-1)(s-3)} = -\dfrac{1}{s-1} + \dfrac{1}{s-3}}$ (*)

Also: $\dfrac{1}{s^2(s-1)(s-3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{s-3}$

or $1 = As(s-1)(s-3) + B(s-1)(s-3) + Cs^2(s-3) + Ds^2(s-1)$

For $s=0$ we get $1 = B(3) \Rightarrow \boxed{B = 1/3}$

$s=1$ we get $1 = C(-2) \Rightarrow \boxed{C = -1/2}$

$s = 3$ $\quad 1 = D(9)(2) \Rightarrow \boxed{D = 1/18}$

$s = 2 \Rightarrow 1 = A(2)(1)(-1) + B(1)(-1) + C(4)(-1) + D(4)$

$1 = -2A - B - 4C + 4D$

$1 = -2A - \frac{1}{3} + 2 + \frac{2}{9} \Rightarrow \boxed{A = \frac{4}{9}}$

so $\boxed{\dfrac{1}{s^2(s-1)(s-3)} = \dfrac{4}{9}\cdot\dfrac{1}{s} + \dfrac{1}{3}\cdot\dfrac{1}{s^2} - \dfrac{1}{2}\cdot\dfrac{1}{s-1} + \dfrac{1}{18}\cdot\dfrac{1}{s-3}}$ (**)

$\mathcal{L}^{-1}\left[\dfrac{2}{(s-1)(s-3)} + \dfrac{1}{s^2(s-1)(s-3)}\right] = [\text{from (*) and (**)}]$

$y(t) = \mathcal{L}^{-1}\left[\dfrac{4}{9}\cdot\dfrac{1}{s} + \dfrac{1}{3}\cdot\dfrac{1}{s^2} - \dfrac{3}{2}\cdot\dfrac{1}{s-1} + \dfrac{19}{18}\cdot\dfrac{1}{s-3}\right] \Rightarrow$

$\boxed{y(t) = \frac{4}{9} + \frac{t}{3} - \frac{3}{2}e^t + \frac{19}{18}e^{3t}}$

b) (b1): homogeneous equation:

$y'' - 4y' + 3y = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0 \qquad \lambda_1 = 1, \lambda_2 = 3$

$y_1 = e^t,\ y_2 = e^{3t} \Rightarrow$ FSS: $\{e^t, e^{3t}\}$

$y_H = c_1e^t + c_2e^{3t}$

(b2) particular solution:

$y_p(t) = At + B \Rightarrow \quad y_p = \frac{1}{3}t + \frac{4}{9}$

(b3) general solution:

$y_G(t) = c_1e^t + c_2e^{3t} + At + B$

(b4) Solution to the IVP:

$\begin{cases} c_1 + c_2 + \frac{4}{9} = 0 \\ c_1 + 3c_2 + \frac{1}{3} = 2 \end{cases} \Rightarrow c_1 = -\frac{3}{2};\ c_2 = \frac{19}{18}$, as above.

(4) a) $\mathcal{L}^{-1}\left(\frac{1}{(s^2+4)(s+1)}\right) = \cos(2t) + \frac{1}{2}\sin(2t)$

hint

Write $\frac{1}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{c}{s+1}$

b) $Y(s) = \frac{e^{-s+5}}{s^2+6s+10} = \frac{e^{-s+5}}{(s+3)^2+1} = \frac{e^{-(s+3)+8}}{(s+3)^2+1}$

$\Rightarrow \mathcal{L}^{-1}(Y(s)) = \boxed{H(t-1)\sin(t-1)\,e^{-3t+8}}$

#5 see also example 5.2, 5.3, 5.9 in the textbook.

a) $f(t) = \begin{cases} 4, & 0 \le t < 2 \\ -1, & 2 \le t < 4 \\ 2, & 4 \le t \end{cases}$

$f(t) = 4\,H_{02}(t) + (-1)\,H_{24}(t) + 2\,H_4(t)$

$f(t) = 4\,[H(t) - H(t-2)] - [H(t-2) - H(t-4)] +$
$+ 2H(t-4) \Rightarrow$

$\boxed{f(t) = 4H(t) - 5H(t-2) + 3H(t-4)}$

b) $f(t) = \begin{cases} t+1; & 0 \le t < 1 \\ t; & 1 \le t \end{cases}$

$f(t) = (t+1)\,H_{01}(t) + t\,H_1(t) =$

$= (t+1)\,[H(t) - H(t-1)] + t\,H(t-1)$

$\boxed{f(t) = (t+1)H(t) - H(t-1)}$

Problems 6, 7 - done in class