M340 Final Exam, *Monday, 12 December*
Fall 2005 - Instructor: Dr. Iuliana Oprea

NAME: ____________________

Clearly answer each of the questions below. Show your work and any formulas you employ. Evaluate any integrals which can be solved by elementary means, and simplify your answers as far as possible. **Box your answers.** Grading will be also based on orderly and transparent presentation.

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1. (20 points)

Determine if the equation

$$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

is exact or not. If the equation is exact, find the general solution. If not, check if an integrating factor $\mu(x)$ or $\mu(y)$ for this equation exists or not and, in the affirmative case, use it to find the general solution of the equation.

2. (30 points) Consider the initial value problem

$$t^3y' + 4t^2y = e^{-t},\ \ y(-1) = 0.$$

(a) What is the interval on which the solution exists?

(b) Find the solution.

3. (20 points)

For the following equations find a particular solution using the method of the undetermined coefficients and write down the general solution. [**XC 5 pts:** choose one of the two equations and solve the IVP $y(0) = 1, y'(0) = 1$ associated to it].

a) $y'' - 2y' - 3y = e^{-t}$

b) $y'' + 4y = 3\sin(2t)$

4. (20 points) Solve the initial value problem

$$y''' - 4y' = 0,\ \ y(0) = 1,\ \ y'(0) = 1,\ y''(0) = -1$$

5. (30 points)

(a) Use Laplace transform to transform the initial value problem

$$y'' - y = e^{2t},\ \ y(0) = 0,\ \ y'(0) = 1$$

into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

(b) Find the solution of the initial value problem.

6. (20 points)

Consider the homogeneous system

$$x' = \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} x.$$

a) write down the general solution;

b) sketch the phase portrait.

7. (20 points)

A 50-g mass (1 kg = 1000 g) stretches a spring 20 cm (1 m = 100 cm). Find a damping constant γ so that the system is critically damped. If the mass is displaced 15 cm from its equilibrium position and released from the rest, find the position of the mass as a function of time and plot the solution.

8. (40 points)

Classify the type of phase plane portrait of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the following matrices $\mathbf{A}$. Also state whether the origin is stable or unstable. [**XC 5 pts.: Sketch the phase portrait**]

a)

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x},$$

b)

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x},$$

c)

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x},$$

d)

$$\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x},$$

XC (10 points) Consider the equation

$$t^2 y'' - 2y = 2t^3$$

Show that $\{t^2, t^{-1}\}$ is a fundamental set of solutions for the homogeneous equation. Find the general solution of the nonhomogeneous equation.