Isochronicity-induced bifurcations in systems of weakly dissipative coupled oscillators

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Abstract

We consider the dynamics of networks of oscillators that are weakly dissipative perturbations of identical Hamiltonian oscillators with weak coupling. Suppose the Hamiltonian oscillators have angular frequency $\omega(\alpha)$ when their energy is α. We address the problem of what happens in a neighbourhood of where $d\omega/d\alpha = 0$; we refer to this as a point of isochronicity for the oscillators. If the coupling is much weaker than the dissipation we can use averaging to reduce the system to phase equations on a torus. We consider example applications to two and three weakly diffusively coupled oscillators with points of isochronicity and reduce to approximating flows on tori. We use this to identify bifurcation of various periodic solutions on perturbing away from a point of isochronicity.

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1 Introduction

Coupled oscillator systems have long been used as models for a variety of physical applications in fields as diverse as biology and solid mechanics. In addition to direct applications they have proven themselves as highly useful mathematical models for investigating dynamics and bifurcations. Beyond the simplest models of linear or weakly nonlinear systems, most types of coupled oscillator models can only be investigated using numerical methods of simulation or continuation.

In this paper we turn to a class of models that, although they are strongly nonlinear, they are amenable to a thorough analysis. This is the class of oscillator networks that are one degree of freedom Hamiltonian oscillators perturbed by (a) weak dissipation and (b) weak coupling. Such models have been introduced and used by several authors studying weakly dissipative Hamiltonian systems, see for example [13, 7, 11, 10]. By changing to action-angle variables one can parametrise the solutions of the Hamiltonian systems in the usual way and so place the equations into a form where they are amenable to averaging. As the systems are strongly nonlinear they can exhibit, for example, homoclinic and heteroclinic orbits, and bifurcations from these.

Systems of n symmetrically coupled identical oscillators were investigated in [4] under assumptions of weak dissipation and very weak coupling; in these circumstances it is possible to reduce the asymptotic dynamics to a flow on an n torus; hence one can find bifurcations of the system by looking on the n torus. In [4] it was found that the stability of the in-phase oscillation for diffusively coupled oscillators depends fundamentally on the sign of the

anisochronicity $\frac{d\omega}{d\alpha}$ where each oscillator is close to a level curve with energy α and period $\omega(\alpha)$ of the underlying Hamiltonian system. (This result was also found by [1] in a weakly nonlinear context). Hence it is natural question to examine the unfolding of isochronicity in order to understand and predict the qualitiative changes of stability and branching that occurs there, and this forms the motivation for this paper.

If a periodic orbit of the Hamiltonian system satisfies

$$\frac{d\omega}{d\alpha}(\alpha_0) = 0$$

(later on we write $\omega_{,\alpha} = \frac{d\omega}{d\alpha}$) then we say that the oscillator has a point of *isochronicity* at the energy level α_0. Typical analyses (for example [4]) are valid away from such points; in this paper we turn our attention to precisely such points and investigate their unfoldings under the non-degeneracy assumption that

$$\frac{d^2\omega}{d\alpha^2}(\alpha_0) \neq 0.$$

The remainder of the paper is organized as follows: in Section 2 we discuss the oscillators, dissipation and coupling in detail. In Section 3 we change to action-angle variables and average the equations over the fast timescale to result in equations (20). In these coordinates it is possible to find the invariant torus. Note that we are perturbing away from an uncoupled system that is nonhyperbolic in the singular (Hamiltonian) limit and hence any exponential attraction of limit cycles is non-uniform in the parameters and needs to be carefully discussed. Any direct application of methods requiring exponential attraction such as those discussed in [12, 9] is therefore not possible in the singular limit.

The dynamics on this torus is investigated for systems of three coupled oscillators in Section 4 for systems that unfold a point of isochronicity. The predictions in that section are considered in Section 5 for numerical simulations of a model sixth order Hamiltonian oscillator weakly coupled in a group of three by diffusive coupling. Finally, some conclusions are outlined in Section 6.

2 Networks of near-conservative identical oscillators

We consider the dynamics of networks of oscillators governed by ODEs of the form

$$\ddot{v}_i + U'(v_i) = \epsilon F_i\left(v_1, \ldots, v_n, \dot{v}_1, \ldots, \dot{v}_n, \lambda\right) \tag{1}$$

where ϵ is a small parameter, $i = 1 \ldots n$, and U and F_i are smooth in all variables. The parameter λ will be used as an unfolding parameter but at presently can be thought of just as a bifurcation parameter.

We assume U has a local minimum and so almost all trajectories in the limit $\epsilon = 0$ are periodic. This system is a singular perturbation of the following Hamiltonian system for each oscillator,

$$H(v, \dot{v}) = \frac{\dot{v}^2}{2} + U(v) = \alpha. \tag{2}$$

We scale the time on the solution curves of (2), defining a positive function $\omega(\alpha)$ such that all periodic solutions of the equation

$$\frac{\omega^2}{2}\left(\frac{\partial z}{\partial \psi}\right)^2 + U(z) = \alpha \tag{3}$$

have period 2π in ψ. The solutions $z(\psi, \alpha)$ to (3) are unique if the phase dependence is normalized, which is achieved by stipulating that

$$z(\alpha, 0) = \max_\psi \{z(\alpha, \psi)\}.$$

The frequency function $\omega(\alpha)$ is defined on all periodic energy levels in (2) with period $T(\alpha)$ by

$$\omega(\alpha) = 2\pi/T(\alpha).$$

We assume that in the region of energy of interest T is bounded above, i.e. we are away from the neighbourhood of homoclinic or heteroclinic orbits. Generally (away from homoclinic or heteroclinic orbits) T is a smooth function of α and may have minima or maxima on varying α. If $d\omega/d\alpha$ is non-zero at a particular level α_0 the sign of this slope has been shown to determine the stability or otherwise of coupled oscillations in the limit of very weak coupling [4, 7].

In that case that $\omega'(\alpha) = 0$ the oscillators have a point of *isochronicity* at α; otherwise we say α is a point of *anisochronicity*. Note that linear oscillators are degenerate even for Hamiltonian oscillators in that they have constant period on all periodic orbits; all energy levels show isochronicity. To avoid problems caused by such degeneracies, we consider points of isochronicity that are a quadratic maximum or minimum.

On addition of weak dissipation in one oscillator and varying a parameter in the dissipation, non-degenerate points of isochronicity will clearly occur with codimension one; more specifically one can unfold the isochronicity using one parameter.

Coupling and dissipation

In general the functions F_i will consist of a part that is dependent only on x_i and another part that is dependent on the other oscillators and vanishes when all oscillators are in the same state. The first part is referred to as the dissipation and the second as the coupling. To make this explicit we write

$$F_i = f_i + \kappa g_i, \tag{4}$$

where the f_i, g_i are of the form

$$\begin{aligned} f_i &= f(v_i, \dot{v}_i) \\ g_i &= \textstyle\sum_{j \neq i} g(v_j, v_i, \dot{v}_j, \dot{v}_i). \end{aligned}$$

The parameter κ is introduced to enable discussion of differing orders of coupling and dissipation. If κ is sufficiently small, then the dynamics will be slaved to that on an invariant n-torus, whereas if it is of order unity such an invariant torus typically will not exist. As in [4] we focus in this paper on the case of small κ. The case where coupling and dissipation are of the same order will be examined elsewhere.

If there is no coupling, i.e., $\kappa = 0$, the conservative system described by the Hamiltonian is perturbed by dissipative, autonomous terms of order ϵ. Setting $v_i = v$ we look for solutions of equation (1) of the form

$$v(t) = z(\alpha(t), \psi(t)), \tag{5}$$

subject to the constraint

$$\frac{dv}{dt} = \omega(\alpha) \frac{\partial z}{\partial \psi}(\alpha, \psi). \tag{6}$$

For notational convenience (to allow subscripts referring to oscillator number) we write the partial derivatives

$$z_{,\alpha} = \frac{\partial z}{\partial \alpha} \qquad z_{,\psi} = \frac{\partial z}{\partial \psi} \quad \text{etc.}$$

By some elementary manipulations [4, 6, 7] it is possible to write the slow evolution of the angle ψ and the energy α as

$$\dot{\alpha} = \epsilon\omega z_{,\psi} f, \tag{7}$$
$$\dot{\psi} = \omega - \epsilon\omega z_{,\alpha} f. \tag{8}$$

3 Averaged Equations

We now derive averaged equations for the case where there is weak dissipation but the coupling is even weaker, i.e., κ is considered as a small parameter whose order will be specified below. Since we are particularly interested in codimension one phenomena, we assume that f_i and g_i are dependent on another variable λ in a regular way. The change of variables $(v_i, \dot{v}_i) \to (\alpha_i, \psi_i)$, defined by $v_i = z(\alpha_i, \psi_i) \equiv z_i$, $\dot{v}_i = \omega(\alpha_i) z_{,\psi}(\alpha_i, \psi_i) \equiv \omega_i z_{i,\psi}$ leads, for $\kappa \neq 0$, to a coupled system of equations for the energies α_i and angles ψ_i which is obtained by augmenting (7),(8) by the coupling terms. For compactness of notation we write

$$\begin{aligned} \dot{\alpha}_i &= \epsilon(a_i + \kappa b_i) \\ \dot{\psi}_i &= \omega_i - \epsilon(c_i + \kappa d_i) \end{aligned} \tag{9}$$

where

$$\begin{aligned} a_i &= \omega_i z_{i,\psi} f_i \\ b_i &= \omega_i z_{i,\psi} g_i \\ c_i &= \omega_i z_{i,\alpha} f_i \\ d_i &= \omega_i z_{i,\alpha} g_i. \end{aligned} \tag{10}$$

The quantities $\{a_i \dots d_i\}$ are functions of $\{\alpha_j, \psi_j\}$ and the parameter λ (they can also be functions of ϵ but this is not important to leading order). In the case of one oscillator we drop the subscript i.

3.1 Uncoupled oscillators with $\epsilon > 0$.

Firstly we consider an uncoupled oscillator at finite $\epsilon > 0$ and fixed λ. With the above notation the equations (7),(8) become

$$\begin{aligned} \dot{\alpha} &= \epsilon a(\alpha, \psi) \\ \dot{\psi} &= \omega(\alpha) - \epsilon c(\alpha, \psi). \end{aligned} \tag{11}$$

We assume that for small enough ϵ and an open set for λ, the system (11) has a periodic solution with period $P_\epsilon = 2\pi/\Omega_\epsilon$ (which is typically also a function of λ) and apply the method of averaging to find this periodic solution. Thus we seek a solution of the form $\alpha(t) = \alpha_\epsilon(\Omega_\epsilon t)$, $\psi(t) = \psi_\epsilon(\Omega_\epsilon t)$, where the functions

$$\left.\begin{aligned} \alpha_\epsilon(\phi) &= \alpha^{(0)} + \epsilon\alpha^{(1)}(\phi) + \epsilon^2\alpha^{(2)}(\phi) + O(\epsilon^3) \\ \psi_\epsilon(\phi) &= \phi + \epsilon\psi^{(1)}(\phi) + O(\epsilon^2), \end{aligned}\right\} \tag{12}$$

are required to be 2π-periodic in ϕ with $\alpha^{(0)}$ being a constant, and

$$\Omega_\epsilon = \omega(\alpha^{(0)}) + \epsilon\Omega_1 + O(\epsilon^2).$$

By substituting this into (11) we find at first order in ϵ,

$$\left.\begin{aligned} \omega\tfrac{d\alpha^{(1)}}{d\phi} &= a(\phi) \\ \omega\tfrac{d\psi^{(1)}}{d\phi} &= \omega_{,\alpha}\alpha^{(1)} - c(\phi) - \Omega_1, \end{aligned}\right\} \tag{13}$$

and at second order the energy equation becomes

$$\omega \frac{d\alpha^{(2)}}{d\phi} = a_{,\alpha}(\phi)\alpha^{(1)} + a_{,\phi}(\phi)\psi^{(1)} - \Omega_1 \frac{d\alpha^{(1)}}{d\phi}, \tag{14}$$

with all energies evaluated at $\alpha = \alpha^{(0)}$. Integrating the first equation of (13) yields

$$\alpha^{(1)}(\phi) - \alpha^{(1)}(0) = \frac{1}{\omega}\int_0^{\phi} a(s)\, ds\,. \tag{15}$$

We define, for general values of α,

$$A(\alpha) = \frac{1}{\omega(\alpha)}\int_0^{2\pi} a(\alpha,\phi)\, d\phi$$

and assume that $\alpha^{(0)}$ is determined such that

$$A(\alpha^{(0)}) = 0,\ A' \equiv \frac{dA(\alpha^{(0)})}{d\alpha} < 0,$$

which is the usual first order averaging condition that assures the existence of a stable limit cycle solution to (11) for sufficiently small $\epsilon > 0$ [15, 14]. The vanishing average of a implies that (15) defines a 2π-periodic function $\alpha^{(1)}(\phi)$ whose initial value $\alpha^{(1)}(0)$ can be found by second order averaging in the following way [15, 14]. For $\alpha^{(2)}(\phi)$ to be periodic we have to require that the average of the r.h.s. of (14) vanishes. Using a partial integration and employing the second equation in (13) and the zero average of a for $\alpha = \alpha^{(0)}$, this condition can be rewritten as

$$\int_0^{2\pi} [a(\phi)c(\phi) + (a_{,\alpha}(\phi) - \omega_{,\alpha}a(\phi))\alpha^{(1)}(\phi)]\, d\phi = 0.$$

This gives a linear equation which can be solved for $\alpha^{(1)}(0)$,

$$\alpha^{(1)}(0) = \frac{1}{A'}\int_0^{2\pi} \frac{a(\phi)}{\omega}[\int_0^{\phi} \frac{a_{,\alpha}(s)}{\omega}\, ds - c(\phi)]\, d\phi\,,$$

thus $\alpha^{(1)}(\phi)$ is determined completely. The first order frequency correction is then found from the second equation in (13) as

$$\Omega_1 = \omega_{,\alpha}\alpha^{(1)}(0) - \frac{1}{2\pi}\int_0^{2\pi} [\frac{\omega_{,\alpha}}{\omega}\phi a(\phi) + c(\phi)]\, d\phi\,.$$

Thus we have determined 2π-periodic functions $\alpha_\epsilon(\phi)$, $\psi_\epsilon(\phi)$ with asymptotic forms given by (12), in terms of which the stable limit cycle solution of (15) is represented as $\alpha_\epsilon(\Omega_\epsilon t), \psi_\epsilon(\Omega_\epsilon t)$ such that errors are uniformly bounded in time. Note that at a point of isochronicity there are typically no bifurcations of the limit cycle of the uncoupled oscillator. Such a change only becomes significant on coupling several such systems together.

3.2 Coupling the oscillators

Now we consider the coupled system, $\kappa \neq 0$. Specifically we assume that

$$\kappa = \epsilon,$$

i.e., the coupling is one order smaller than the dissipation. The weak coupling suggests to pursue a perturbation analysis about the limit cycle of an unperturbed oscillator. Since

$(\alpha_\epsilon, \psi_\epsilon)$ have been determined as functions of a single angle ϕ, we can change coordinates from (α_i, ψ_i) to (γ_i, ϕ_i) by the transformation

$$(\alpha_i, \psi_i) = (\alpha_\epsilon(\phi_i) + \epsilon\gamma_i, \psi_\epsilon(\phi_i))$$

and look for solutions with γ_i of order $O(1)$. The equations for the evolution of the perturbed energies γ_i and phases ϕ_i are given by

$$\left.\begin{aligned} \alpha'_\epsilon\dot{\phi}_i + \epsilon\dot{\gamma}_i &= \epsilon a_i(\alpha_\epsilon + \epsilon\gamma_i) + \epsilon^2 b_i(\alpha_\epsilon + \epsilon\gamma_i) \\ \psi'_\epsilon\dot{\phi}_i &= \omega_i(\alpha_\epsilon + \epsilon\gamma_i) - \epsilon c_i(\alpha_\epsilon + \epsilon\gamma_i) - \epsilon^2 d_i(\alpha_\epsilon + \epsilon\gamma_i), \end{aligned}\right\} \tag{16}$$

(α'_ϵ, ψ'_ϵ are the derivatives with respect to their arguments). Only arguments with respect to energy are shown; the dependence on ϕ_i and λ has been suppressed (note that $\alpha_\epsilon = \alpha_\epsilon(\phi_i)$). The equation for $\dot{\phi}_i$ can be rewritten as

$$\dot{\phi}_i = \Omega_\epsilon \frac{\omega_i(\alpha_\epsilon + \epsilon\gamma_i) - \epsilon c_i(\alpha_\epsilon + \epsilon\gamma_i)}{\omega_i(\alpha_\epsilon) - \epsilon c_i(\alpha_\epsilon)} - \epsilon^2\Omega_\epsilon \frac{d_i(\alpha_\epsilon + \epsilon\gamma_i)}{\omega_i(\alpha_\epsilon) - \epsilon c_i(\alpha_\epsilon)}. \tag{17}$$

3.3 Unfolding isochronicity for coupled oscillators

Suppose there is a point of non-degenerate isochronicity at $\lambda = 0$. Without loss of generality (by adding a constant to the potential U) we can assume that this occurs at energy $\alpha = 0$ in the uncoupled system. Moreover, writing $A(\alpha, \lambda)$ to make the λ-dependence explicit, we assume that $A' \equiv A_{,\alpha}(0,0) < 0$ so that there is a stable limit cycle for each uncoupled oscillator at energy $\alpha^{(0)}(\lambda)$ ($\alpha^{(0)}(0) = 0$) depending smoothly on λ. In addition, we impose the transversality condition $A_{,\lambda}(0,0) \neq 0$, i.e. $\alpha^{(0)}_{,\lambda}(0) \neq 0$, hence without loss of generality (redefining λ if necessary) we can assume that

$$\alpha^{(0)}(\lambda) = \lambda.$$

Note that by assumption, the uncoupled Hamiltonian oscillators are independent of λ. Generically ω will have a quadratic maximum (or minimum) at 0 and so we write

$$\omega(\alpha) = \omega_0 + K\alpha^2 + O(\alpha^3)$$

for some $K \neq 0$. This situation is clearly robust; small perturbations to the equations (along with suitable rescaling of the coordinates) will merely displace the position of this codimension-one situation. We rescale a neighbourhood of $\lambda = 0$ by writing

$$\lambda = \epsilon\mu$$

with $\mu = O(1)$ (so that there is a stable limit cycle with energy $\alpha^{(0)} = \epsilon\mu$). We also assume henceforth that $\gamma = O(1)$ for all time (this will be shown *a posteriori*) and hence the error terms in the following equations are uniform in t. Thus we obtain

$$\omega_i(\alpha_\epsilon) = \omega_0 + \epsilon^2 K(\mu + \alpha_i^{(1)})^2 + O(\epsilon^3),$$

$$\omega_i(\alpha_\epsilon + \epsilon\gamma_i) = \omega_0 + \epsilon^2 K(\mu + \alpha_i^{(1)} + \gamma_i)^2 + O(\epsilon^3).$$

Moreover we expand

$$c_i(\alpha_\epsilon + \epsilon\gamma_i) = c(\alpha_\epsilon) + \epsilon\gamma_i \frac{dc}{d\alpha}(\alpha_\epsilon) + O(\epsilon^2),$$

and note that $\Omega_\epsilon = \omega_0 + \epsilon\Omega_1 + O(\epsilon^2)$, where now

$$\Omega_1 = -\frac{1}{2\pi}\int_0^{2\pi} c(\phi)\, d\phi,$$

with c evaluated at $\lambda = 0$. Substituting these expansions into (17) and expanding in powers of ϵ yields

$$\dot{\phi}_i = \Omega_\epsilon + \epsilon^2 \left(K(\mu + \alpha_i^{(1)} + \gamma_i)^2 - K(\mu + \alpha_i^{(1)})^2 - \frac{dc_i}{d\alpha}\gamma_i - d_i(0) \right) + O(\epsilon^3).$$

Note that there are several sources for second order terms in this equation. This is used to eliminate the $\dot{\phi}_i$ term from the equations for $\dot{\gamma}_i$, i.e. (see (14))

$$\epsilon\dot{\gamma}_i = \epsilon a_i(\alpha_\epsilon + \epsilon\gamma_i) + \epsilon^2 b_i(\alpha_\epsilon + \epsilon\gamma_i) - \epsilon\dot{\phi}_i a_i(\alpha_\epsilon)/\Omega_\epsilon. \tag{18}$$

Expanding in powers of ϵ gives

$$\epsilon\dot{\gamma}_i = \epsilon a_i + \epsilon^2 \frac{da_i}{d\alpha}\gamma_i - \epsilon a_i + \epsilon^2 b_i + O(\epsilon^3), \tag{19}$$

where all the functions are evaluated at $\alpha = \alpha_\epsilon$. Cancelling a factor of ϵ in the γ_i equation leads finally to the following equations for the very weakly coupled, weakly dissipative system,

$$\left.\begin{array}{rcl} \dot{\gamma}_i & = & \epsilon \left(\frac{da_i}{d\alpha}\gamma_i + b_i \right) + O(\epsilon^2) \\ \dot{\phi}_i & = & \Omega_\epsilon + \epsilon^2 \left(2K(\mu + \alpha_i^{(1)})\gamma_i - \frac{dc_i}{d\alpha}\gamma_i - d_i \right) + O(\epsilon^3). \end{array}\right\}, \tag{20}$$

where now, noting that $\alpha_\epsilon = O(\epsilon)$ means $a_i(\alpha_\epsilon) = O(\epsilon)$, all terms are evaluated at energy $\alpha = 0$.

The system (20) provides the generalization of equation (25) in [4], which are valid near generic values of $\alpha^{(0)}$, to the codimension one situation defined by $\omega_{,\alpha}(\alpha^{(0)}) = 0$. While the γ_i-equations are the same as in [4], the main difference occurs in the ϕ-equations which

(i) have nontrivial terms at $O(\epsilon^2)$ but not at $O(\epsilon)$,

(ii) depend on the $O(1)$-parameter μ which describes the passage through isochronicity

(iii) depend on γ_i as well as on ϕ_i.

3.4 Averaged equations and slow manifold

The assumption that the Hamiltonian oscillators and dissipation functions are identical imply that our system is close to a $1:1:\cdots:1$ resonance and so we introduce a total phase

$$\Theta = \frac{1}{n}\sum_{i=1}^{n} \phi_i,$$

and $n-1$ complementary angles whose time variations are $O(\epsilon^2)$. A convenient choice is

$$\theta_i = \phi_{i+1} - \phi_i \quad (1 \le i \le n-1),$$

although in the presence of symmetries appropriate linear combinations of the θ_i may be more adequate (see section 4). In these coordinates (20) takes the form

$$\left.\begin{array}{rcl} \dot{\gamma}_i & = & \epsilon(\frac{da_i}{d\alpha}\gamma_i + b_i) + O(\epsilon^2) \\ \dot{\theta}_i & = & \epsilon^2 H_i + O(\epsilon^3) \\ \dot{\Theta} & = & \omega_0 + \epsilon\Omega_1 + O(\epsilon^2), \end{array}\right\} \tag{21}$$

where

$$H_i = [2K(\mu + \alpha_{i+1}^{(1)}) - \frac{dc_{i+1}}{d\alpha}]\gamma_{i+1} - [2K(\mu + \alpha_i^{(1)}) - \frac{dc_i}{d\alpha}]\gamma_i - d_{i+1} + d_i\,,$$

and b_i, d_i etc. are now considered as functions of $(\Theta, \theta_1, \ldots, \theta_{n-1})$. The system (21) is in the appropriate form for the averaging procedure (although there are different orders in ϵ, it is easy to define a near identity transformation which shifts fluctuations about Θ-averages to higher order). We write

$$\langle b_i \rangle = \frac{1}{2\pi} \int_0^{2\pi} b_i(\Theta, \theta_1, \ldots, \theta_{n-1}) d\Theta$$

to denote the average of b_i and analogously for other averaged quantities. Noting that the average of $da_i/d\alpha$ is just $\omega_0 A'$ with A' as defined before, the averaged equations truncated at leading order then read

$$\left.\begin{array}{rcl} \dot{\gamma}_i & = & \epsilon(\omega_0 A' \gamma_i + \langle b_i \rangle) + O(\epsilon^2) \\ \dot{\theta}_i & = & \epsilon^2 \langle H_i \rangle + O(\epsilon^3). \end{array}\right\} \tag{22}$$

Since $A' < 0$ the system (22) evolves on an invariant ("superslow") $(n-1)$-torus. To leading order we can use adiabatic elimination to determine the vector field governing the superslow motion. Thus we can set $\gamma_i = -\langle b_i \rangle / \omega_0 A'$ (justifying the assumption that the γ_i remain $O(1)$) and obtain

$$\dot{\theta}_i = \epsilon^2 (\Lambda B_i - D_i) + O(\epsilon^3), \tag{23}$$

where

$$B_i = \langle b_{i+1} - b_i \rangle \,, \quad D_i = \langle d_{i+1} - d_i \rangle \tag{24}$$

are smooth functions of $(\theta_1, \ldots, \theta_{n-1})$, and

$$\Lambda = -\frac{1}{\omega_0 A'} [2K(\mu + \alpha^{(1)}(0)) - \langle \frac{dc}{d\alpha} \rangle] \tag{25}$$

is considered as new bifurcation parameter (taking $O(1)$ values). Together with the overall Θ-motion, (23) describes the dynamics on an invariant n-torus and its fixed points correspond to periodic orbits on that torus. Note that hyperbolic fixed points of (22) are in one-one correspondence with hyperbolic periodic orbits of (21). Moreover, one can easily show that the eigenvalues of the Jacobian of (23) at a fixed point consist of n eigenvalues given by $\epsilon A' + O(\epsilon^2) < 0$ and $n-1$ further eigenvalues of order $O(\epsilon^2)$ whose leading terms are fully determined by the Jacobian of (23).

Various types of bifurcations may occur when Λ is varied, depending on the specific forms of the B_i, D_i. In particular one can expect symmetry breaking steady state bifurcations as well as saddle nodes for (23) when the couplings show symmetries.

In Section 4 we apply this reduction to examine two and three oscillators near isochronicity with permutation symmetries in more detail; Section 5 examines a specific example of this analysis and compares it to some numerical simulations. In the following we drop the subscript 0 on ω and identify ω with its value ω_0 at isochronicity.

4 Networks with permutation symmetries

4.1 Two coupled oscillators

We consider first the case of two coupled oscillators with an exchange symmetry

$$(v_1, \dot{v}_1, v_2, \dot{v}_2) \to (v_2, \dot{v}_2, v_1, \dot{v}_1)$$

in the coupling functions. Owing to this symmetry we can write $b_1 = b(\phi_1, \phi_2)$, $b_2 = b(\phi_2, \phi_1)$ and analogously $d_1 = d(\phi_1, \phi_2)$, $d_2 = d(\phi_2, \phi_1)$, with single coupling functions b, d. The system (23) then reduces to a single equation for $\theta \equiv \theta_1 = \phi_2 - \phi_1$ which can be written as

$$\dot{\theta} = \epsilon^2 [\Lambda B(\theta) - D(\theta)] + O(\epsilon^3), \tag{26}$$

where

$$D(\theta) = \langle d(\Theta + \theta/2, \Theta - \theta/2) - d(\Theta - \theta/2, \Theta + \theta/2)\rangle, \tag{27}$$

and $B(\theta)$ is given analogously in terms of b.

Since both $B(\theta)$ and $D(\theta)$ are odd functions of θ, (26) always admits the steady states $\theta_+ = 0$ (in phase solution) and $\theta_- = \pi$ (out of phase solution). Generically the derivatives $B'(\theta)$ and $D'(\theta)$ will be nonzero, thus we find pitchfork bifurcations of these steady states at $\Lambda = D'(\theta_\pm)/B'(\theta_\pm)$. In addition, saddle node bifurcations of asymmetric steady states may occur at points $\Lambda = D(\theta_0)/B(\theta_0)$ whenever $D'(\theta_0)B(\theta_0) = D(\theta_0)B'(\theta_0)$. Thus we conclude that all bifurcations that one generically finds in symmetric flows on a circle, can be expected to occur near a change of sign of isochronicity in two coupled, weakly dissipative oscillators for which the coupling is weaker than the dissipation.

Linear diffusive coupling. This type of coupling is described by

$$g_1 = r(v_2 - v_1), \quad g_2 = r(v_1 - v_2). \tag{28}$$

The functions B, D take here the form of convolutions,

$$B(\theta) = \omega r\langle z_{,\psi}(\Theta)[z(\Theta - \theta) - z(\Theta + \theta)]\rangle \tag{29}$$

$$D(\theta) = \omega r\langle z_{,\alpha}(\Theta)[z(\Theta - \theta) - z(\Theta + \theta)]\rangle, \tag{30}$$

with all quantities evaluated at $\alpha = \alpha^{(0)}$ (we now drop the subscript on ω). Since $\omega_{,\alpha}(\alpha^{(0)}) = 0$, we may pick any phase shift in $z(\psi)$ when computing the averages (29),(30), thus we can write

$$z(\psi) = v(\psi/\omega), \quad z_{,\psi}(\psi) = \frac{1}{\omega}\dot{v}(\psi/\omega), \quad z_{,\alpha}(\psi) = v_{,\alpha}(\psi/\omega). \tag{31}$$

Here, $v(t)$ is a solution of the unperturbed Hamiltonian (2) which we choose to be even, $v(t) = v(T - t)$, where $T = 2\pi/\omega$ is the period. Accordingly $v_{,\alpha}(t)$ is also even whereas $\dot{v}(t)$ is odd, $\dot{v}(t) = -\dot{v}(T - t)$. Note that $v_{,\alpha}(t)$ is a periodic function only if $\omega_{,\alpha} = 0$, for other energies $z_{,\alpha}(\psi)$ is a linear combination of $\dot{v}$ and $v_{,\alpha}$. At isochronicity the two convolutions in (29) have opposite signs and those in (30) have equal signs. Setting $t_\theta = \theta/\omega$ we thus can write

$$\begin{aligned} B(\theta) &= -2r\omega\langle z_{,\psi}(\Theta)z(\Theta + \theta)\rangle \\ &= -\frac{r\omega}{\pi}\int_0^{T/2} \dot{v}(t)[v(t + t_\theta) - v(t - t_\theta)]dt \end{aligned} \tag{32}$$

$$\begin{aligned} B'(\theta) &= -2r\omega\langle z_{,\psi}(\theta)z_{,\psi}(\Theta + \theta)\rangle \\ &= -\frac{r}{\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}(t + t_\theta) + \dot{v}(t - t_\theta)]dt\,, \end{aligned} \tag{33}$$

whereas $D(\theta) = 0$. Consequently the bifurcations at all for this kind of coupling have degenerate branching behaviour. Any fixed point is a zero of $B(\theta)$ and generically the derivative $B'(\theta)$ at such a zero will be nonzero. When Λ passes through zero, all stability coefficients simply change their signs without giving rise to the disappearance or creation of new fixed points (vertical bifurcations). Moreover, if there is an additional reflection symmetry, i.e., $U(-v) = U(v)$, and if the oscillation has a zero mean, we also encounter a symmetry in the half period, $v(T/2 - t) = -v(t)$ and $\dot{v}(T/2 - t) = \dot{v}(t)$. In this case we find

$$B'(\pi) = \frac{2r}{\pi}\int_0^{T/2} \dot{v}(t)\dot{v}(t + T/2)dt = -\frac{4r}{\pi}\int_0^{T/4} \dot{v}^2(t)dt = -B'(0)\,, \tag{34}$$

hence at $\kappa = 0$ the stabilities of the in phase and out of phase solutions are reversed, as was observed already in [4] in the analysis of the generic case (away from a point of isochronicity).

Linear diffusive coupling in the derivatives. For this kind of coupling the coupling functions are

$$g_1 = s(\dot{v}_2 - \dot{v}_1), \quad g_2 = s(\dot{v}_1 - \dot{v}_2). \tag{35}$$

Similarly as before we find here that

$$\begin{aligned} D(\theta) &= -2s\omega^2 \langle z_{,\alpha}(\Theta) z_{,\psi}(\Theta+\theta)\rangle \\ &= -\frac{s\omega^2}{\pi}\int_0^{T/2} v_{,\alpha}(t)[\dot{v}(t+t_\theta) - \dot{v}(t-t_\theta)]dt \qquad (36) \\ D'(\theta) &= -2s\omega^2 \langle z_{,\alpha}(\theta) z_{,\psi\psi}(\Theta+\theta)\rangle \\ &= -\frac{s\omega}{\pi}\int_0^{T/2} v_{,\alpha}(t)[\ddot{v}(t+t_\theta) + \ddot{v}(t-t_\theta)]dt\,, \qquad (37) \end{aligned}$$

whereas now $B(\theta) = 0$, thus the roles of B and D are interchanged. Since the bifurcation parameter does not enter the averaged equations, no bifurcations are expected to occur near isochronicity for this coupling. As before, if $U(-v) = U(v)$ and if the oscillation has zero mean, the stabilities of the in phase and out of phase solutions are opposite, since in this case

$$D'(\pi) = \frac{2s}{\pi}\int_0^{T/2} \dot{v}(t)\dot{v}_{,\alpha}(t+T/2)dt = -\frac{4s}{\pi}\int_0^{T/4} \dot{v}(t)\dot{v}_{,\alpha}(t)dt = -D'(0)\,. \tag{38}$$

General couplings. It is obvious that the above considerations hold also for nonlinear coupling functions of the form $g_1 = h(v_2) - h(v_1)$ and $g_1 = h(\dot{v}_2) - h(\dot{v}_1)$. In the first case we just have stability exchanges of the fixed points at $\kappa = 0$ and in the second case no stability exchanges can be expected. While the latter certainly corresponds to generic behaviour, the "pure" stability exchange in the first case is highly ungeneric and should be attributed to the special form of the coupling function. For more general couplings involving both v_i and $\dot{v}_i$, we can expect generic pitchforks from the in phase and out of phase solutions as well as generic saddle nodes of asymmetric fixed points. This occurs already for linear couplings of the form

$$g_1 = r(v_2 - v_1) + s(\dot{v}_2 - \dot{v}_1) \quad \text{with} \quad rs \neq 0, \tag{39}$$

for which $B(\theta)$ and $D(\theta)$ are given by (32) and (36), respectively. Analogous integral representations can be derived for higher derivatives of B and D, which are needed at bifurcation points.

Representation as Abelian integrals. For the two types of linear coupling discussed the functions B, D are represented by convolution averages involving $z, z_{,\psi}, z_{,\alpha}, z_{,\psi\alpha}$. We can rewrite these convolutions in terms of Abelian integrals related to the Hamiltonian of a single, unperturbed oscillator. Here v is considered as independent variable with range

$$v_-(\alpha) \leq v \leq v_+(\alpha), \tag{40}$$

where $v_\pm(\alpha)$ are the turning points defined by $U(v_\pm) = \alpha$. Let

$$X(v,\alpha) = \sqrt{2(\alpha - U(v))} \geq 0 \tag{41}$$

be the velocity during the first half period and define a "time function"

$$\tau(v,\alpha) = \int_{v_-}^{v} \frac{dv}{X}, \tag{42}$$

i.e., $0 \le \tau \le T/2$ when v varies through its range (40), hence $T(\alpha) = 2\tau(v_+)$. To any phase shift θ we associate a shift function $v_\theta(v,\alpha)$ implicitely by the relation

$$\pm\tau(v_\theta) = \tau(v) + \frac{\theta T}{2\pi} \mod T\,, \tag{43}$$

with the right hand side projected to the range $-T/2 \le t \le T/2$. Finally let

$$X_\theta(v,\alpha) = \pm\, X(v_\theta,\alpha) \;\; (X_0 = X) \tag{44}$$

be the velocity along the time shifted closed orbit with the sign as defined in (43).

Noting that $v_{,\alpha} = -\dot{v}\tau_{,\alpha}$ during the first half period, the t-integrals occurring in the averages introduced before are readily converted to Abelian integrals, yielding (at $\alpha = \alpha^{(0)}$),

$$\langle z_{,\psi}(\Theta) z(\Theta+\theta)\rangle = \frac{1}{2\pi}\int_{v_-}^{v_+} (v_\theta - v_{-\theta})\, dv\,, \tag{45}$$

$$\langle z_{,\alpha}(\Theta) z(\Theta+\theta)\rangle = -\frac{\omega}{2\pi}\int_{v_-}^{v_+} \tau_{,\alpha}(v_\theta + v_{-\theta})\, dv\,, \tag{46}$$

$$\langle z_{,\psi}(\Theta) z_{,\psi}(\Theta+\theta)\rangle = -\frac{1}{2\pi\omega}\int_{v_-}^{v_+} (X_\theta + X_{-\theta})\, dv\,, \tag{47}$$

$$\langle z_{,\alpha}(\Theta) z_{,\psi}(\Theta+\theta)\rangle = -\frac{1}{2\pi}\int_{v_-}^{v_+} \tau_{,\alpha}(X_\theta - X_{-\theta})\, dv\,, \tag{48}$$

$$\langle z_{,\alpha}(\Theta) z_{,\psi\psi}(\Theta+\theta)\rangle = \frac{1}{2\pi\omega}\int_{v_-}^{v_+} \tau_{,\alpha}[U(v_\theta) + U(v_{-\theta})]\, dv\,. \tag{49}$$

Since each of these functions is either even or odd, evaluation over a half period is sufficient. In a similar, straightforward manner one can express averages that occur in higher derivatives of B and D as Abelian integrals.

4.2 Three coupled oscillators

Now we consider three oscillators coupled bidirectionally in a ring with identical nearest neighbour couplings. The system has D_3-symmetry, the symmetry of the dihedral group of six elements. Accordingly, the coupling functions b_i, d_i are represented by single functions $b(\phi_1,\phi_2,\phi_3)$, $d(\phi_1,\phi_2,\phi_3)$ which are invariant under exchange of the second and third arguments, $b(\phi_1,\phi_2,\phi_3) = b(\phi_1,\phi_3,\phi_2)$, as

$$b_i = b(\phi_i,\phi_j,\phi_k), \;\; d_i = d(\phi_i,\phi_j,\phi_k), \tag{50}$$

where (i,j,k) is a permutation of $(1,2,3)$. As in subsection 3.3 we set $\Theta = (\phi_1+\phi_2+\phi_3)/3$. Instead of using the relative phases $\theta_1 = \phi_2 - \phi_1$, $\theta_2 = \phi_3 - \phi_2$, it is here more convenient to utilize the complex equivariant projection

$$\theta \equiv \xi + \imath\eta = \phi_1 + \rho\phi_2 + \rho^2\phi_3, \tag{51}$$

where $\rho = e^{2\pi\imath/3}$ is the primitive cube root of unity. Thus we have

$$\xi = \theta_1 - \frac{1}{2}\theta_2\,, \quad \eta = -\frac{\sqrt{3}}{2}\theta_2\,,$$

and

$$\phi_i = \Theta + \frac{1}{3}(\bar{\rho}^{i-1}\theta + \rho^{i-1}\bar{\theta})\,.$$

Given any real valued function $f(\phi_1, \phi_2, \phi_3)$ which is 2π-periodic in each argument and possesses the reflection invariance $f(\phi_1, \phi_2, \phi_3) = f(\phi_1, \phi_3, \phi_2)$, we define its symmetrized Θ-average,

$$f_{av}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\Theta + \xi, \Theta + \frac{\eta}{\sqrt{3}}, \Theta - \frac{\eta}{\sqrt{3}})\, d\Theta\,. \tag{52}$$

This function is periodic with respect to the hexagonal lattice generated by 2π and $2\pi\rho$ in the θ-plane, and reflection invariant, $f_{av}(\bar{\theta}) = f_{av}(\theta)$. We then define the full average of f by an additional average over the symmetry group,

$$f_{AV}(\theta) = f_{av}(\theta) + \bar{\rho} f_{av}(\rho\theta) + \rho f_{av}(\bar{\rho}\theta), \tag{53}$$

which is also periodic on the hexagonal lattice and in addition D_3-equivariant,

$$f_{AV}(\rho\theta) = \rho f_{AV}(\theta), \quad f_{AV}(\bar{\theta}) = \bar{f}_{AV}(\theta). \tag{54}$$

Owing to this equivariance and the generation through averages, one readily verifies that $f_{AV}(\theta_0) = 0$ for

$$\left.\begin{array}{lll} \theta_0 = 0 : & \text{in phase} & (\theta_1 = \theta_2 = 0) \\ \theta_0 = \pm 2\pi\imath/\sqrt{3} : & \text{rotating wave} & (\theta_1 = \theta_2 = \pm 2\pi/3). \end{array}\right\} \tag{55}$$

Moreover, $f_{AV}(\theta)$ is real when $\theta = \xi \in \mathbf{R}$ by virtue of the reflection invariance.

After the transformation (51) the system (23) is transformed (in complex notation) to

$$\dot{\theta} = \epsilon^2[\Lambda B(\theta) - D(\theta)] + O(\epsilon^3), \tag{56}$$

where $D(\theta) = d_{AV}(\theta)$, $B(\theta) = b_{AV}(\theta)$. The system (56) has fixed points $\theta = \theta_0$ as in (55). In addition, since the real (ξ) axis is invariant, we generically find real fixed points $\theta_0 = \xi_0 \in \mathbf{R}$ in saddle node bifurcations. These fixed points correspond to pair synchronized solutions ($\eta = 0$, i.e. $\theta_2 = 0$). Asymmetric fixed points and periodic orbits (with large periods) can of course also exist, depending on the specific forms of B and D. Note that periodic orbits correspond to quasiperiodic solutions of the unaveraged system. Since (56) is a generic one-parameter system, all kinds of codimension one bifurcations encountered in two dimensional vector fields leading to such solutions, in particular bifurcations from the symmetric solutions (55), can be expected to occur.

The stabilities of the symmetric solutions (55) and $\theta_0 = \xi_0$ are governed by the Jacobians of B and D. We therefore have to study the form of the eigenvalues and eigenvectors at such points for vector fields $f_{AV}(\theta)$ obtained by averaging a reflection invariant function $f(\phi_1, \phi_2, \phi_3)$ as described before. Let $f_{,i} = \partial f/\partial\phi_i$ $(i = 1, 2, 3)$ and set

$$\begin{aligned} f_{,s}(\theta) &= \langle f_{,1}(\Theta + \xi, \Theta + \eta/\sqrt{3}, \Theta - \eta/\sqrt{3})\rangle, \\ f_{,a}(\theta) &= \langle f_{,2}(\Theta + \xi, \Theta + \eta/\sqrt{3}, \Theta - \eta/\sqrt{3}) - f_{,2}(\Theta + \xi, \Theta - \eta/\sqrt{3}, \Theta + \eta/\sqrt{3})\rangle. \end{aligned}$$

These functions are again periodic on the hexagonal lattice and $f_{,s}$ is even while $f_{,a}$ is odd under $\theta \to \bar{\theta}$. The derivatives of f_{AV} at a general point θ can then be written in terms of $f_{,s}, f_{,a}$ as

$$2\frac{\partial f_{AV}(\theta)}{\partial\theta} = f_{,s}(\theta) + f_{,s}(\bar{\rho}\theta) + f_{,s}(\rho\theta) - \frac{\imath}{\sqrt{3}}[f_{,a}(\theta) + f_{,a}(\bar{\rho}\theta) + f_{,a}(\rho\theta)]\,, \tag{57}$$

$$4\frac{\partial f_{AV}(\theta)}{\partial\bar{\theta}} = 2f_{,s}(\theta) - f_{,s}(\bar{\rho}\theta) - f_{,s}(\rho\theta) + \frac{2\imath}{\sqrt{3}}[\rho f_{,a}(\rho\theta) + \bar{\rho} f_{,a}(\bar{\rho}\theta)]\,. \tag{58}$$

Specializing (57),(58) to the distinguished fixed points θ_0 yields the following forms for the Jacobians $df_{AV}(\theta_0)$ at these points:

(i) The Jacobian at the in phase solution, $\theta_0 = 0$, is proportional to the identity,

$$2\, df_{AV}(0) = 3\, f_{,s}(0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{59}$$

(ii) In case of the rotating waves, $\theta_0 = \pm 2\pi\imath/\sqrt{3}$, we find $\partial f_{AV}/\partial\bar{\theta} = 0$, hence there is generically a complex conjugate pair of eigenvalues, given by

$$2\frac{\partial f_{AV}}{\partial\theta}(\pm\frac{2\pi\imath}{\sqrt{3}}) = 3f_{,s}(\frac{2\pi\imath}{\sqrt{3}}) \mp \imath\sqrt{3}f_{,a}(\frac{2\pi\imath}{\sqrt{3}})\,. \tag{60}$$

(iii) Finally, at a real fixed point $\theta = \xi_0$, invariance of the real line implies that there are real eigenvalues $\lambda_\xi(f)$ and $\lambda_\eta(f)$ with eigendirections along the real and imaginary axes, respectively, given by

$$2\lambda_\xi(f) = 2f_{,s}(\xi_0, 0) + f_{,s}(-\xi_0/2, \xi_0/2) - f_{,a}(-\xi_0/2, \xi_0/2)\,, \tag{61}$$

$$2\lambda_\eta(f) = 3f_{,s}(-\xi_0/2, \xi_0/2) + f_{,a}(-\xi_0/2, \xi_0/2)\,. \tag{62}$$

When (59),(60) are evaluated for b, d, we find generically nonzero derivatives for (59) and nonzero real and imaginary parts for (60). Thus when Λ is varied, the in phase solution will encounter a D_3-symmetric steady state bifurcation and the rotating waves will undergo a Hopf bifurcation giving rise to quasiperiodicity in the original system. In addition, there are generically at least two saddle node bifurcations leading to pair oscillations $\theta = \xi_0$ which may or may not, depending on the specific forms of b, d, undergo pitchfork bifurcations when the η-eigenvalue pases through zero.

Linear diffusive coupling. We consider again linear diffusive coupling which is here described by

$$g_1 = r(v_2 + v_3 - 2v_1), \tag{63}$$

thus the functions $b(\phi_1, \phi_2, \phi_3), d(\phi_1, \phi_2, \phi_3)$ are given by

$$b = r\omega z_{,\psi}(\phi_1)[z(\phi_2) + z(\phi_3) - 2z(\phi_1)], \tag{64}$$

$$d = r\omega z_{,\alpha}(\phi_1)[z(\phi_2) + z(\phi_3) - 2z(\phi_1)]. \tag{65}$$

$$\tag{66}$$

Using (52),(53) it is straightforward to find expressions for $B(\theta) = b_{AV}(\theta)$ and $D(\theta) = d_{AV}(\theta)$ in terms of the solutions $v(t)$ of the original Hamiltonian system (2). Since these are lengthy, we present only their restrictions to $\theta = \xi \in \mathbf{R}$. As in the preceding subsection the phases are chosen such that $v(t)$ is even in $0 \le t \le T$. We obtain

$$\begin{aligned} B(\xi) &= r\omega\langle z_{,\psi}(\Theta)[2z(\Theta - \xi) - z(\Theta + \xi)]\rangle \\ &= \frac{3r\omega}{2\pi}\int_0^{T/2} \dot{v}(t)[v(t - t_\theta) - v(t + t_\theta)]\, dt\,, \end{aligned} \tag{67}$$

$$\begin{aligned} D(\xi) &= r\omega\langle z_{,\alpha}(\Theta)[2z(\Theta - \xi) - z(\Theta + \xi) - z(\Theta)]\rangle \\ &= \frac{r\omega^2}{2\pi}\int_0^{T/2} v_{,\alpha}(t)[v(t + t_\theta) - 2v(t) + v(t - t_\theta)]\, dt\,, \end{aligned} \tag{68}$$

where $t_\xi = \xi T/2\pi$. Note that $B(\pi) = 0$, however $D(\pi) \neq 0$ in general. The eigenvalues $\lambda_\xi(f), \lambda_\eta(f)$ summarized in (61),(62) can, for $f = b$ and $f = d$, be expressed analogously as

$$\begin{aligned}
\lambda_\xi(b) &= -3r\omega\langle z_{,\psi}(\Theta) z_{,\psi}(\Theta+\xi_0)\rangle \\
&= -\frac{3r}{2\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}(t+t_{\xi_0}) + \dot{v}(t-t_{\xi_0})]dt, \qquad (69)\\
\lambda_\eta(b) &= -2r\omega\langle z_{,\psi}(\Theta)[z_{,\psi}(\Theta+\xi_0) + 2z_{,\psi}(\Theta)]\rangle \\
&= -\frac{r}{\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}(t+t_{\xi_0}) + \dot{v}(t-t_{\xi_0}) + 4\dot{v}(t)]dt, \qquad (70)\\
\lambda_\xi(d) &= -r\omega\langle z_{,\psi}(\Theta)[z_{,\alpha}(\Theta+\xi_0) + z_{,\alpha}(\Theta-\xi_0)]\rangle \\
&= -\frac{r\omega}{2\pi}\int_0^{T/2} \dot{v}(t)[v_{,\alpha}(t+t_{\xi_0}) - v_{,\alpha}(t-t_{\xi_0})]dt = -\lambda_\eta(d). \qquad (71)
\end{aligned}$$

When $\xi_0 = 0$ these eigenvalues reduce to $\lambda_\xi(d) = \lambda_\eta(d) = 0$, and

$$\lambda_\xi(b) = \lambda_\eta(b) = -\frac{3r}{\pi}\int_0^{T/2} \dot{v}^2(t)dt. \qquad (72)$$

Hence, as in the case of two coupled oscillators, the in phase solution reverses stability when Λ passes through zero and this stability reversal is not combined with the creation of bifurcating asymmetric solutions, as in the case of two coupled oscillators. In contrast, for pair oscillations we can expect generic saddle nodes and pitchforks in and off the real line, respectively.

We still need the eigenvalues for the rotating waves. Specializing (60) for b and d we readily find

$$\begin{aligned}
\frac{\partial B}{\partial\theta}(\pm\frac{2\pi\imath}{\sqrt{3}}) &= -\frac{3}{2}r\omega\langle z_{,\psi}(\Theta)[z_{,\psi}(\Theta+\frac{2\pi}{3}) + z_{,\psi}(\Theta-\frac{2\pi}{3})]\rangle \\
&= -\frac{3r}{2\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}(t+T/3) + \dot{v}(t-T/3)]dt\,, \qquad (73)\\
\frac{\partial D}{\partial\theta}(\pm\frac{2\pi\imath}{\sqrt{3}}) &= \pm\imath\frac{\sqrt{3}}{2}r\omega\langle z_{,\psi}(\Theta)[z_{,\alpha}(\Theta+\frac{2\pi}{3}) - z_{,\alpha}(\Theta-\frac{2\pi}{3})]\rangle \\
&= \pm\imath\frac{\sqrt{3}}{\pi}r\omega\int_0^{T/2} \dot{v}(t)[v_{,\alpha}(t+T/3) - v_{,\alpha}(t-T/3)]dt\,. \qquad (74)
\end{aligned}$$

Just as in the case of two oscillators we find degenerate behaviour: there is just a stability reversal of the rotating waves when Λ passes through zero, but no other branches of solutions are created at this instability. For nondegenerate coupling we expect generic Hopf bifurcations leading to quasiperiodicity in the original system (see, e.g., [2]).

Linear diffusive coupling in the derivatives. Here we have

$$g_1 = s(\dot{v}_2 + \dot{v}_3 - 2\dot{v}_1), \qquad (75)$$

and $B(\xi), D(\xi)$ take the form

$$\begin{aligned}
B(\xi) &= s\omega^2\langle z_{,\psi}(\Theta+\xi) z_{,\psi}(\Theta) - z_{,\psi}^2(\Theta)\rangle \\
&= \frac{s\omega^3}{2\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}(t+t_\xi) + \dot{v}(t-t_\xi) - 2\dot{v}(t)]dt\,, \qquad (76)\\
D(\xi) &= s\omega^2\langle z_{,\alpha}(\Theta)[2z_{,\psi}(\Theta-\xi) - z_{,\psi}(\Theta+\xi)]\rangle \\
&= \frac{3s\omega^3}{2\pi}\int_0^{T/2} v_\alpha(t)[\dot{v}(t-t_\xi) - \dot{v}(t+t_\xi)]dt\,. \qquad (77)
\end{aligned}$$

Hence for this coupling $D(\pi) = 0$ whereas $B(\pi) \neq 0$ in general. The eigenvalues at fixed points $\theta = \xi_0$ are

$$\begin{aligned}
\lambda_\xi(b) &= \frac{s}{2}\omega^2\langle z_{,\psi}(\Theta)[3z_{,\psi\psi}(\Theta+\xi_0) + z_{,\psi\psi}(\Theta-\xi_0)]\rangle \\
&= \frac{s}{2\pi}\int_0^{T/2} \dot{v}(t)[\ddot{v}(t+t_{\xi_0}) - \ddot{v}(t-t_{\xi_0})]dt = -\lambda_\eta(b) \qquad (78)\\
\lambda_\xi(d) &= s\omega^2\langle z_{,\psi}(\Theta)[2z_{,\alpha\psi}(\Theta+\xi_0) + z_{,\alpha\psi}(\Theta-\xi_0)]\rangle \\
&= \frac{3s\omega}{2\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}_{,\alpha}(t+t_{\xi_0}) + \dot{v}_{,\alpha}(t-t_{\xi_0})]dt \qquad (79)\\
\lambda_\eta(d) &= s\omega^2\langle z_{,\psi}(\Theta)[z_{,\alpha\psi}(\Theta-\xi_0) + 2z_{,\alpha\psi}(\Theta)]\rangle \\
&= \frac{s\omega}{2\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}_{,\alpha}(t+t_{\xi_0}) + \dot{v}_{,\alpha}(t-t_{\xi_0}) + 4\dot{v}_{,\alpha}(t)]dt\,, \qquad (80)
\end{aligned}$$

and for $\xi_0 = 0$ we have

$$\lambda_\xi(d) = \lambda_\eta(d) = \frac{3s\omega}{\pi}\int_0^{T/2} \dot{v}(t)\dot{v}_{,\alpha}(t)dt\,. \qquad (81)$$

Hence there is no bifurcation of the in-phase solution whereas saddle nodes of two-in-phase solutions will generically occur. Observe that the roles of b and d in the eigenvalue structure are reversed for the two types of diffusive couplings, as in the case of two coupled oscillators. For the rotating waves we obtain

$$\begin{aligned}
\frac{\partial B}{\partial\theta}(\pm\frac{2\pi\imath}{\sqrt{3}}) &= \pm\imath\frac{\sqrt{3}}{2}s\omega^2\langle z_{,\psi}(\Theta)[z_{,\psi\psi}(\Theta+\frac{2\pi}{3}) - z_{,\psi\psi}(\Theta-\frac{2\pi}{3})]\rangle \\
&= \pm\imath\frac{\sqrt{3}s}{2\pi}\int_0^{T/2} \dot{v}(t)[\ddot{v}(t+T/3) - \ddot{v}(t-T/3)]dt \qquad (82)\\
\frac{\partial D}{\partial\theta}(\pm\frac{2\pi\imath}{\sqrt{3}}) &= \frac{3}{2}s\omega^2\langle z_{,\psi}(\Theta)[z_{,\alpha\psi}(\Theta+\frac{2\pi}{3}) + z_{,\alpha\psi}(\Theta-\frac{2\pi}{3})]\rangle \\
&= \frac{3s\omega}{2\pi}\int_0^{T/2} \dot{v}(t)[\dot{v}_{,\alpha}(t+T/3) + \dot{v}_{,\alpha}(t-T/3)]dt\,, \qquad (83)
\end{aligned}$$

which implies that no bifurcation from rotating waves will generically occur in the perturbation analysis for this type of coupling.

General linear coupling. The analysis above shows that three symmetrically coupled oscillators show analogous behaviour as was observed in the case of two coupled oscillators: the symmetric (in phase and rotating wave) oscillations encounter vertical bifurcations for linear diffusive coupling whereas no stability exchanges occur for linear diffusive coupling in the derivatives. That this holds also for nonlinear couplings of these types is obvious as in the case of two oscillators. In contrast, the pair oscillations will generically undergo pitchfork and saddle node bifurcations for both couplings. Hence again the bifurcation behaviour of the symmetric solutions is ungeneric and should be attributed to the special forms of the couplings. Generic bifurcations can only be expected near changes of isochronicity when mixed couplings are present, for example, for coupling functions of the form

$$g_1 = r(v_2 + v_3 - 2v_1) + s(\dot{v}_2 + \dot{v}_3 - 2\dot{v}_1)\,. \qquad (84)$$

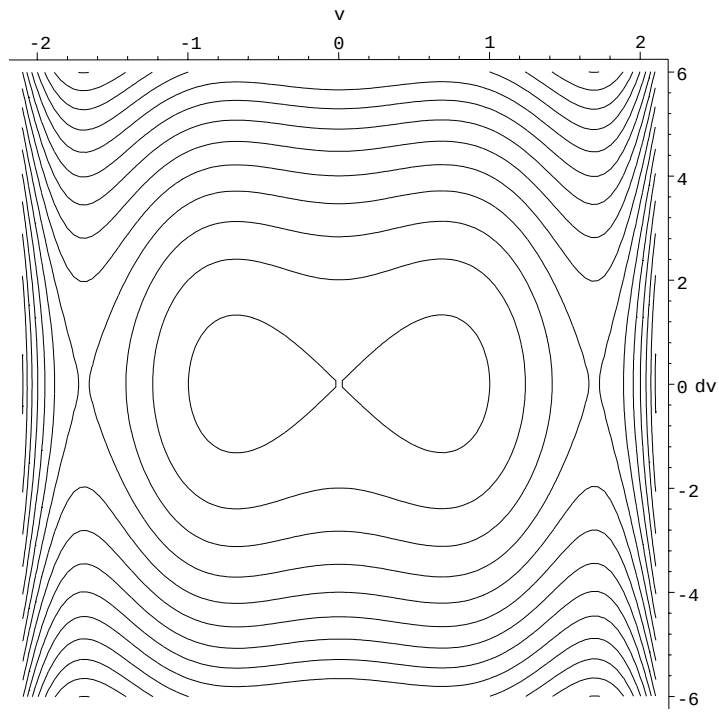

Figure 1: Level curves of the Hamiltonian for a single oscillator with potential (85) in the $(v, \dot{v})$ plane. Note the presence of two homoclinic orbits to the origin and a heteroclinic cycle around this. For the closed orbits between these two levels there is a point of isochronicity corresponding to a maximum of the frequency ω.

5 Example: A Sixth Order Potential

As an example we consider a network whose individual, undamped oscillators are governed by

$$U(v) = -v^2(1 - v^2)(4 - v^2). \tag{85}$$

For this potential the classical motion (2) shows a pair of homoclinic orbits (ambiclinic orbit) at $\alpha = 0$, a pair of heteroclinic orbits at $\alpha = (70 + 26\sqrt{13})/27 \approx 6.065$ and a family of symmetric (zero mean) oscillations between these two values of α. Hence there is at least one point of isochronicity in that range of energies and numerical computation [4] indicates that there is exactly one such change. Figure 1 shows the level curves of the Hamiltonian function $U(v) + \dot{v}^2/2 = constant$; note the presence of five equilibria on the v axis at $v = 0$, ± 0.68177 and ± 1.69367.

For the dissipation function we consider

$$f(v, \dot{v}) = (\beta - v^4)\dot{v}, \tag{86}$$

and treat β as global bifurcation parameter. In the following we discuss first basic properties of a single oscillator. Then we investigate networks of three coupled oscillators as described in the preceeding section unfolding a point of isochronicity.

5.1 Single Oscillator

Let $v^2 = w$ and consider

$$Q(w, \alpha) = \alpha + w(1-w)(4-w). \tag{87}$$

In the range of interest, $0 < \alpha < 6.065$, Q has three real roots $w_1 > w_2 > 0 > w_4$ and the turning points ($\dot{v} = 0$) of the zero mean oscillation occur at $v = \pm\sqrt{w_2}$. We set $x = w_2$ and use x instead of α as independent parameter, with range $1 < x < (5+\sqrt{13})/3 \approx 2.869$. The other roots w_1, w_4 and α are represented parametrically as

$$w_{1,4} = \frac{1}{2}(5 - x \pm R) \;\text{ where }\; R = \sqrt{9 + 10x - 3x^2}\,, \quad \alpha = -4x + 5x^2 - x^3\,.$$

The (odd) solution to $\ddot{v} + U'(v) = 0$ for energy $\alpha(x)$ is given by

$$v(t) = \frac{\sqrt{\alpha}y(t)}{\sqrt{4M^2 - w_1 w_2 y^2(t)}}, \tag{88}$$

where

$$y(t) = \mathrm{sn}(\sqrt{8}Mt, k), \;\; k^2 = \frac{w_2(w_1 - w_4)}{w_1(w_2 - w_4)} = \frac{xR}{4M^2}$$

$$M^2 = \frac{1}{4}w_1(w_2 - w_4) = \frac{1}{8}(xR - 8 + 15x - 3x^2),$$

and $\mathrm{sn}(\eta, k)$ is the standard Jacobi elliptic function of modulus k. The period $T(x)$ (or $T(\alpha)$) can then be expressed in terms of $K(k)$, Jacobi's complete elliptic integral of the first kind,

$$T = \frac{\sqrt{2}}{M}K(k).$$

The parameter value where isochronicity occurs is determined by $MK_{,k} = KM_{,x}k_{,x}$. Using a well-known relation for the derivatives of complete elliptic integrals, this equation can be rewritten as

$$\frac{(2x^2 - 5x - 3)(3x^2 - 15x + 8 + xR)}{5x^3 - 25x^2 + 20x + 12} + 2\frac{E(k)}{K(k)} = 0\,,$$

where $E(k)$ is Jacobi's complete elliptic integral of the second kind. The numerical solution is $x = x_0 \approx 2.0815$ with corresponding energy $\alpha^{(0)} \approx 4.3190$ and frequency $\omega \approx 2.5892$. Computing the second derivative of T at $\alpha^{(0)}$ yields $K = \omega_{,\alpha\alpha} \approx -0.1257$ and so this point of isochronicity is nondegenerate.

Next we determine the value β for which a periodic orbit at $\alpha^{(0)}$ persists the dissipation. Redefining $A(\alpha)$ by $\omega A(\alpha)$, this function is here given by

$$A(\alpha) = \sqrt{8}(\beta A_1(\alpha) - A_2(\alpha)),$$

where

$$A_1 = \int_0^{w_2} \sqrt{\frac{Q}{w}}dw, \quad A_2 = \int_0^{w_2} w^{3/2}\sqrt{Q}dw.$$

Evaluation at $\alpha^{(0)}$ yields $A_1 \approx 5.6680$, $A_2 \approx 3.5281$, hence $\beta_0 = A_2/A_1 \approx 0.6225$ is the persistence value of β for $\alpha = \alpha^{(0)}$. The derivatives A_1', A_2' with respect to α are

$$A_1' = \frac{1}{2}\int_0^{w_2} \frac{dw}{\sqrt{Qw}} = \frac{\pi}{\sqrt{2}\omega}, \quad A_2' = \frac{1}{2}\int_0^{w_2} w^{3/2}\frac{dw}{\sqrt{Q}}$$

with values $A_1' \approx 0.8580$, $A_2' \approx 1.2808$ at $\alpha = \alpha^{(0)}$. This gives $A' = \sqrt{8}(\beta_0 A_1' - A_2') \approx -2.1121$ and so, if we define λ as

$$\lambda = -\frac{A_1}{A'}(\beta - \beta_0) \approx 2.6836(\beta - 0.6225)\,,$$

to leading order in ϵ we have an attracting limit cycle with isochronicity at $\lambda = 0$. To find an expression for our $O(1)$ bifurcation parameter Λ, we first note that $\langle ac \rangle = 0$ because $z_{,\alpha} z_{,\alpha\psi}$ is odd, hence

$$\alpha^{(1)}(0) = -\frac{1}{\omega^2 A'} \int_0^{2\pi} a_{,\alpha}(\phi) \int_0^{\phi} a(s)\, ds\, d\phi,$$

with $a(\phi) = \dot{v}^2(\phi/\omega)(\beta_0 - v^4(\phi/\omega))$. The functions $a(\phi)$, $a_{,\alpha}(\phi)$ can be derived symbolically from the representation (88) which then allows us to compute the integral numerically, giving $\alpha^{(1)}(0) \approx -10.3178$. Since $\langle dc/d\alpha \rangle = 0$, all quantities that occur in (25) have been computed and our $O(1)$–bifurcation parameter Λ can be approximately written as

$$\Lambda \approx 1.2278 + 0.3193(0.6225 - \beta)/\epsilon\,.$$

5.2 Three Oscillators

We have performed some numerical simulations of three coupled oscillators of the type

$$\ddot{v}_1 + U'(v_1) = \epsilon(\beta - v_1^4)\dot{v}_1 + \epsilon^2(r(v_2 + v_3 - 2v_1) + s(\dot{v}_2 + \dot{v}_3 - 2\dot{v}_1))$$

with the other equations obtained by cyclic permutation of the indices and $U(v)$ the sixth order potential defined by (85). For the simulations we use the dynamical systems program `xppaut` [8] and consider $\epsilon = 0.5$ and a range of β, r and s.

In the case that $s = 0$ and $r = 1$ (coupling only via linear diffusive coupling) we find that the rotating wave is stable for $\beta >\sim 0.62$ and the in-phase oscillation is stable for $\beta <\sim 0.62$. It is very hard to resolve how stability is transferred from one to the other, and this is in agreement with the observation that to lowest order, the bifurcation for such coupling will be "vertical". Figure 2 shows the time series for a stable rotating wave of the three oscillators with $\beta = 0.68$. Figure 3 shows the time series for a stable in-phase solution of the three oscillators with $\beta = 0.68$. Figure 4 shows that near the point of isochronicity, $\beta = 0.62$, a small perturbation to the in-phase solution can be attracted to the in-phase solution after undertaking a large homoclinic excursion along a set of two-in-phase solutions. This indicates that the loss of stability is via a degenerate form of the S_3 transcritical homoclinic bifurcation [2].

On coupling with purely diffusive coupling of derivatives, i.e. $r = 0$ and $s \neq 0$ we were not able to find attractors that were more than very slowly attracting; for example with $s = 0.2$ it was possible to find stable in-phase solutions on both sides of $\beta = 0.62$.

For mixed coupling, e.g. $r = 0.4$ and $s = 0.1$ it was possible to see connecting branches, for example stable two-in-phase solutions at $\beta = 0.7$; see Figure 5(a); for $r = -0.5$, $s = 0.1$ and $\beta = 0.54$ we were able to find stable 2-tori that branch from the rotating wave solution at a Hopf bifurcation; see Figure 5(b).

6 Conclusions

We show that isochronicity (i.e. turning points in the frequency of oscillation) of a class of nonlinear coupled oscillators is responsible for a variety of bifurcation effects on unfolding this codimension one degeneracy to a generic situation. Using averaging we obtain equations valid for unfoldings of isochronous oscillators in terms of the solutions of the Hamiltonian

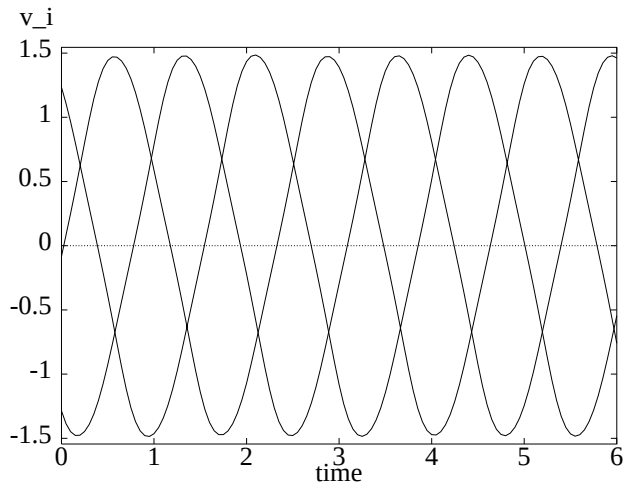

Figure 2: A stable rotating wave for the three oscillator system with $\beta = 0.68$, $r = 1$ and $s = 0$. The three timeseries for v_1, v_2 and v_3 are shown on the same graph and show a 120 degree phase shift between any two.

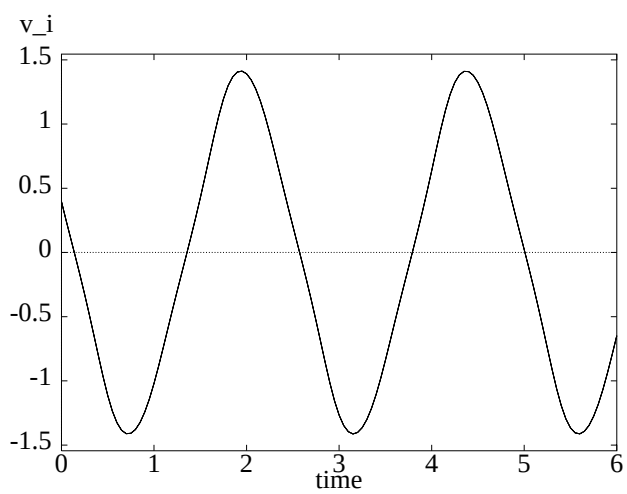

Figure 3: A stable in-phase solution for the three oscillator system with $\beta = 0.58$, $r = 1$ and $s = 0$. The three timeseries for v_1, v_2 and v_3 are shown on the same graph but are indistinguishable.

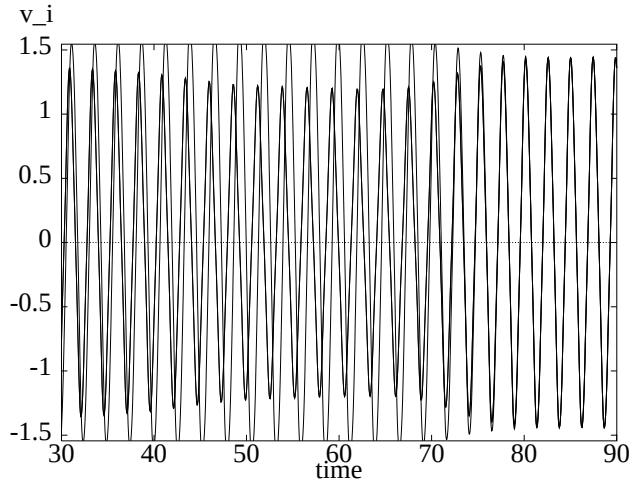

Figure 4: Evidence of a homoclinic connection from in-phase to in-phase solution for the three oscillator system with $\beta = 0.62$, $r = 1$ and $s = 0$. The three timeseries for v_1, v_2 and v_3 start and end near an in-phase solution, but for intermediate times they stay near a two-in-phase solution.

problem and the coupling and dissipation of the oscillators. In particular cases one can solve the Hamiltonian problem using, for example elliptic integrals and then the averaged system is explicitly known.

For the cases of linear diffusive coupling we find that bifurcations do occur, but these will be vertical. For linear diffusive coupling via the derivatives there is typically no bifurcation; only when both forms of coupling are present do we expect non-degenerate branching at an unfolding of a point of isochronicity. This has been verified in a particular system of three coupled weakly dissipative, weakly coupled oscillators with a sixth order potential. This seemingly degenerate behaviour of linear diffusive coupling of derivatives is very intriguing and we suspect that a higher order expansion is necessary to determine the generic behaviour of the branch that is vertical to the order we have determined.

Throughout we have assumed that coupling occurs at an order of magnitude less than the dissipation; this is important to ensure that all asymptotic dynamics occurs on an invariant torus that is close to a product of limit cycles of the Hamiltonian system. If coupling and dissipation occur weakly but at the same order of magnitude, there can be very complicated dynamics of higher dimensional form in the singular limit and the description of the asymptotic dynamics purely in terms of the phases breaks down.

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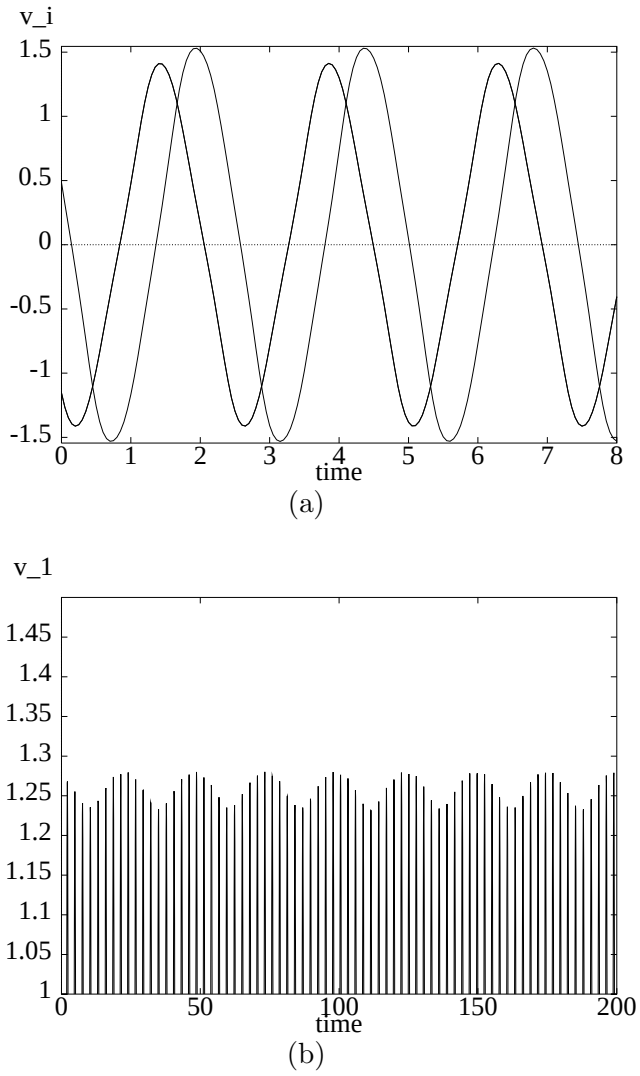

Figure 5: (a) Stable two-in-phase solution for the three oscillator system with $\beta = 0.7$ and mixed coupling $r = 0.4$ and $s = 0.1$. The timeseries for v_1 and v_2 are indistinguishable while v_3 is visibly separate. (b) Stable 2-frequency quasiperoidicity of a rotating wave solution at $r = -0.5$, $s = 0.1$ and $\beta = 0.54$. This manifests itself as a slow modulation of the peak amplitude of one of the oscillators, v_1 in this case. The other two oscillators are approximately 120 degrees out of phase relative to this one.

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