Analysis for Freshman is Impossible You Say?

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Generalities

Principle 1

Set a difficult mathematic/scientific goal that requires tools from analysis and make attaining this goal the focus of the effort

The development and understanding of the tools becomes a sideshow of the solution of the main problem

Mastery of analysis is achieved by employing the tools to address the main goal

This contrasts to the approach of developing a systematic presentation of analysis and gaining mastery by solution of abstract problems before turning to application

Principle 2

Focus the material on the core ideas in analysis

The theoretical emphasis should be on sequences and limits, approximation and convergence, construction of numbers, functions, root and fixed point problems, linearization, solution of differential equations and integration

The practical emphasis should be on using these ideas to solve broad scientific problems

Much of the standard “computational” content of a traditional Calculus course is omitted

Students pick up this kind of material later on their own or in engineering courses by working off a solid theoretical understanding

Principle 3

Use a constructive approach to analysis

All major proofs should be constructive in nature, employing an algorithm that can be implemented on a computer

Establish the practice of students implementing algorithms used in proofs and testing the conclusions of theorems

In several cases, this requires a little more structure than the most general assumptions, e.g., treating Lipschitz continuity rather than pointwise continuity

Principle 4

Coursework consisting of problem sets, computational projects, and in-class exams should emphasize theoretical understanding and application to the solution of practical problems

The problem sets cover completion and extension of proofs and analytic applications

The computational projects require a report detailing data, methodology, analysis, and conclusions

In-class exams cover knowledge of hypotheses and meaning of results

The importance of exact answer solutions in applications and as an indication of understanding is de-emphasized

Organizing Theme

The organizing theme is the use of mathematics to create and understand models of physical phenomena

The role of mathematics in scientific and engineering modeling is described as

- Providing a language for describing physical phenomena
- *A priori* determination of properties of models and solutions, e.g., existence, uniqueness, domain and range
- Computing approximate solutions of mathematical models along with conveying some *a priori* or *a posteriori* information about the accuracy

Binding Threads

Inadequacy

A common trick is to explore the inadequacy of a given - usually comfortably familiar - mathematical system as a way of motivating the construction of a more complicated system

In number systems, we move from

$$\mathbb{N} \longrightarrow \mathbb{I} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R}$$

Inadequacies can be exposed through

- The construction of models
  For example, the first models might use integer valued parameters, but it is easy to find models that require rational valued parameters

- The solution of models
  For example, models that have irrational solutions

Inadequacy

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This can be illustrated through considering patterns - and lack of patterns - in decimal expansions of numbers

The reals are constructed using the notion of Cauchy sequences

This is linked to the use of the Bisection Algorithm for computing roots with the understanding that there is no reason to expect a nice pattern out of the related decimal expansions

Inadequacy

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In functions, we move from

$$\text{linear} \longrightarrow \text{polynomial} \longrightarrow \text{rational} \longrightarrow \text{general}$$

Emphasize the role of polynomials in approximating general functions, akin to the role of rational numbers in approximating reals

The elementary functions are developed as commonly occurring general functions

Experimental Knowledge to Mathematical Conclusions

We observe that almost all direct experience with mathematics is based on a few “data” points and that we are often in the situation of drawing mathematical conclusions by extrapolating from this scanty knowledge

Examples include

- Induction from the first few terms in an iteration
- Computing decimal expansions and long division of polynomials
- Plotting
- Determining information about solutions of differential equations

Experimental Knowledge to Mathematical Conclusions

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This raises the fundamental issues

- Why do we think we have good data points?

  This is largely an issue of convergence and approximation

- Why can we interpolate/extrapolate from these data points?

  This is largely an issue of smoothness of the mathematical process, which here usually means continuity properties of functions

Experimental Knowledge to Mathematical Conclusions

We observe that almost all direct experience with mathematics is based on a few “data” points and that we are often in the situation of drawing mathematical conclusions by extrapolating from this scanty knowledge

A good example is developing properties of real numbers

We can use e.g. the Bisection Algorithm to solve a model for a real number and prove that the solution “solves” the model

Extending the notion of a solution leads to the question of how to compute with such numbers and the development of the properties of real numbers

There is a natural progression to extending function domains to real numbers and plotting

Practical Points

View the definition of a limit as the ability to specify a desired accuracy from an approximation

Reduce the emphasis on the actual limit

Use Lipschitz continuity rather than pointwise continuity

The uniformity simplifies analysis and allows constructive proofs

Estimating and analytic proofs often reduce to finding bounds on a function over an interval

Fixed Point Iteration

The Fixed Point Iteration should occupy a central role

It is the primary tool for approximating solutions of nonlinear models

It provides a practical application of convergence of sequences and the notions of linearization

Application to the inverse and implicit function theorems is developed

It provides the means to develop a very accessible yet highly detailed development of Newton's method and modifications

Calculus

Calculus

Calculus is developed after a relatively long delay

It's introduction is motivated by

- The need to explore properties of models (functions)
- The need to formulate and solve models whose solutions are functions

Linearization and the Derivative

Linearization is developed as the basic approximation and description tool of Calculus

This ties naturally to Lipschitz continuity and generalizes to several dimensions

The derivative is presented as the matrix of the linear transformation with related interpretations following

The approximation of functions is emphasized

An early application to the analysis of the stability of an equilibrium point of an iterated map is presented

The analysis reduces to rewriting a given expression in terms of differences that can be estimated by linearization and finding bounds on functions

The Mean Value Theorem

The Mean Value Theorem as an expression of the lowest order Taylor polynomial approximation is emphasized

It would be difficult to overstates its importance in computational mathematics

We provide a constructive proof that gives a way to compute the "mysterious" point which represents one of the major analytic hurdles

Modeling with Differential Equations

There needs to be some development of the use of differential equations in modeling

“Galileo’s model” of a falling body provides a classic example of creating and solving a model whose solution is a function

Antidifferentiation and formal integration are developed as a systematic way to guess a solution

Minimal substitution and integration by parts are the only formal integration techniques that are covered

Integration and Solving Differential Equations

Integration is developed as a numerical method for approximately solving the simplest differential equation - with direct extension to nonlinear differential equations

The definition and treatment of Cauchy sequences of functions is introduced right at the start

The Fundamental Theorem of Calculus is presented as an approximation and convergence result for the numerical solution algorithm

Area under curves, length of a curve, average values of functions, etcetera are presented as applications

The connection to the Lebesgue theory remains to be developed fully

Elementary Functions

The elementary functions are introduced as solutions of fundamental models

- Model of rocket propulsion $\leftrightarrow$ the logarithm
- Constant relative rate of change models $\leftrightarrow$ the exponential
  Also make the connection to the inverse of the logarithm
- Mass-spring systems $\leftrightarrow$ the trigonometric functions

Some References

R. Courant and F.John, *Introduction to Calculus and Analysis*, Springer-Verlag

D. Estep, *Practical Analysis in One Variable*, Springer-Verlag