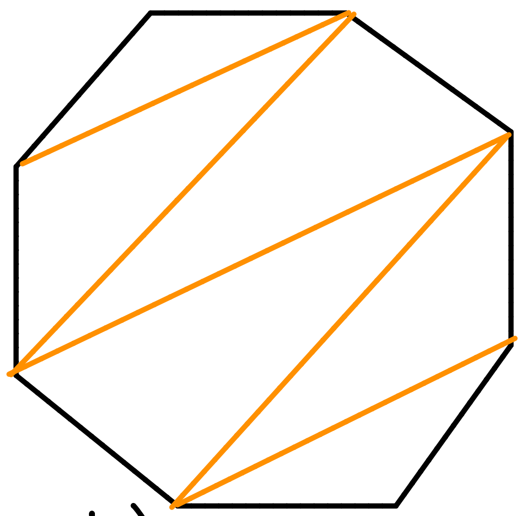
Math 6760: Days 32-33

We saw last time that, provided we have a way of assigning coords to the torus fibers of the moment polytope P^n of a toric symplectic mfd $(M^{2n}, \omega, T^n, \mu)$, we get a measure-preserving map $T^n \times P^n \to M^{2n}$, where we use the std product measure on $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$ & Lebesgue measure on P^n.

In other words, from the perspective of sampling or numerical integration, M is indistinguishable from $T^n \times P^n$, which is in general much easier to deal with.

So, finally, an (almost) toric structure on $Pol(n)$.

First of all, pick a triangulation of an abstract n-gon; e.g.,



Now, the edgelengths are all 1, so knowing the lengths $d_1, \ldots, d_{n-3}$ of the chords uniquely determines the triangles up to congruence (by SSS from high school geometry).

Now, we know from the triangulation which sides of which triangles are glued together, but we have a circle worth of possible ways of performing these gluings, parametrized by the dihedral angles b/w the glued-together triangles.

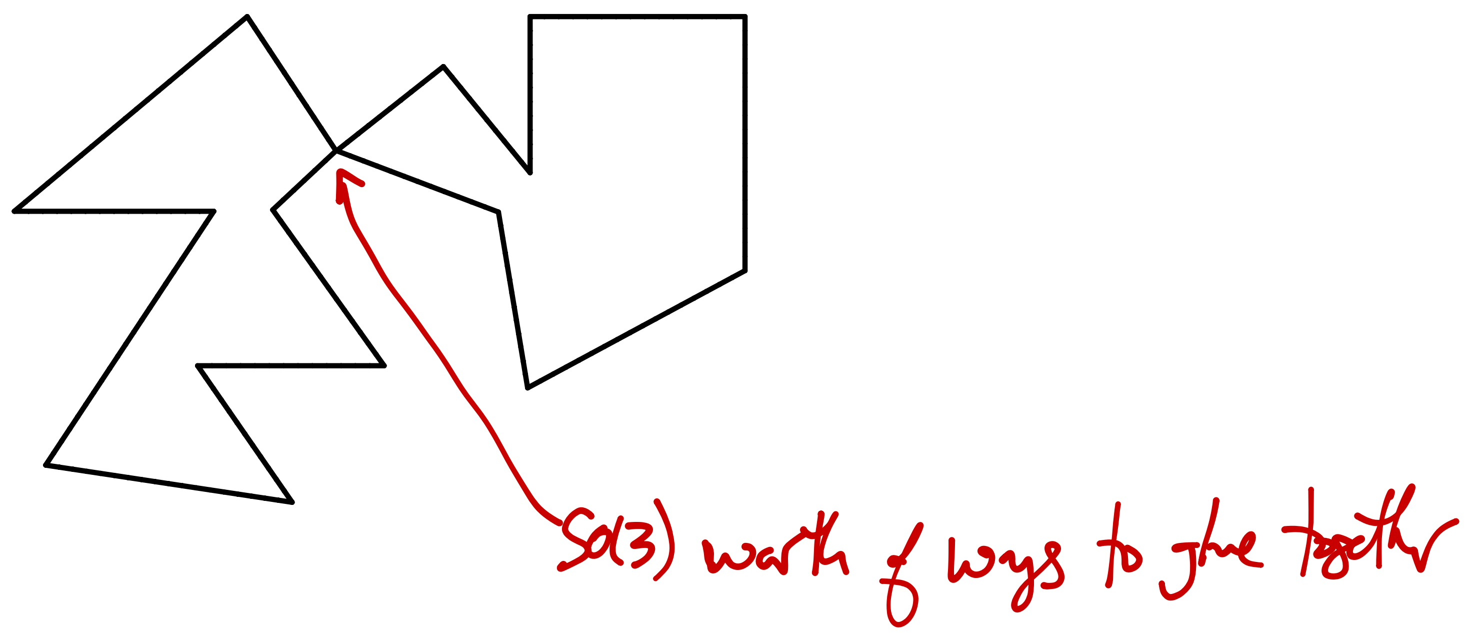
But these are a bunch of circle actions that turn out to be Hamiltonian; since the chords in the triangulation don't intersect, the circle actions commute, so this actually determines a torus action.

Now, the conserved quantity under rotation around the k^{th} chord is the length of the k^{th} chord, so the moment map μ records the chordlengths.

There are $n-3$ chords, so this is an $(n-3)$-dimensional torus. Since $\dim(Pol(n)) = 2n-6$, the dimensions are right for a toric mfd.

Of course, the S^1 action corresponding to the k^{th} chord doesn't make sense when the k^{th} chord has length 0;

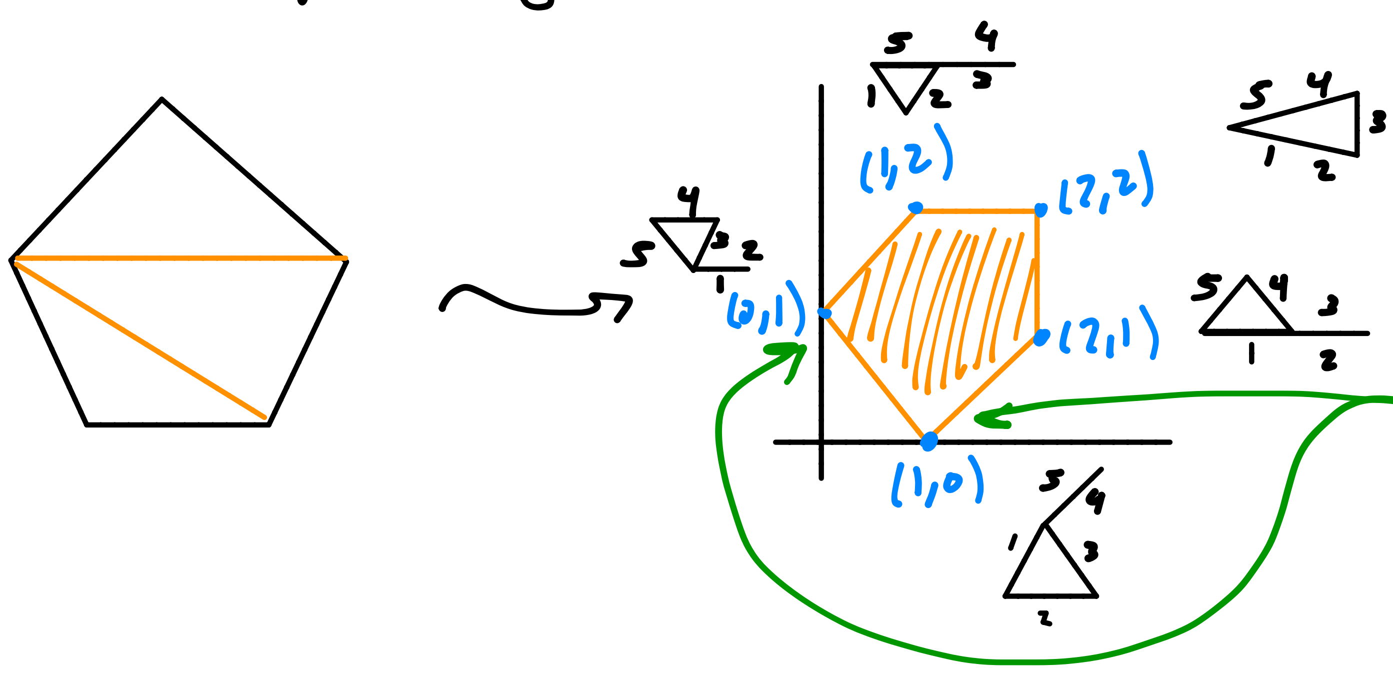
indeed, there are unexpected extra degrees of freedom at these loci:



So this is not exactly toric, though the space of generic configurations certainly is, & since the bad pieces are codim 3, that's all we need to care about from a measure-theoretic perspective.

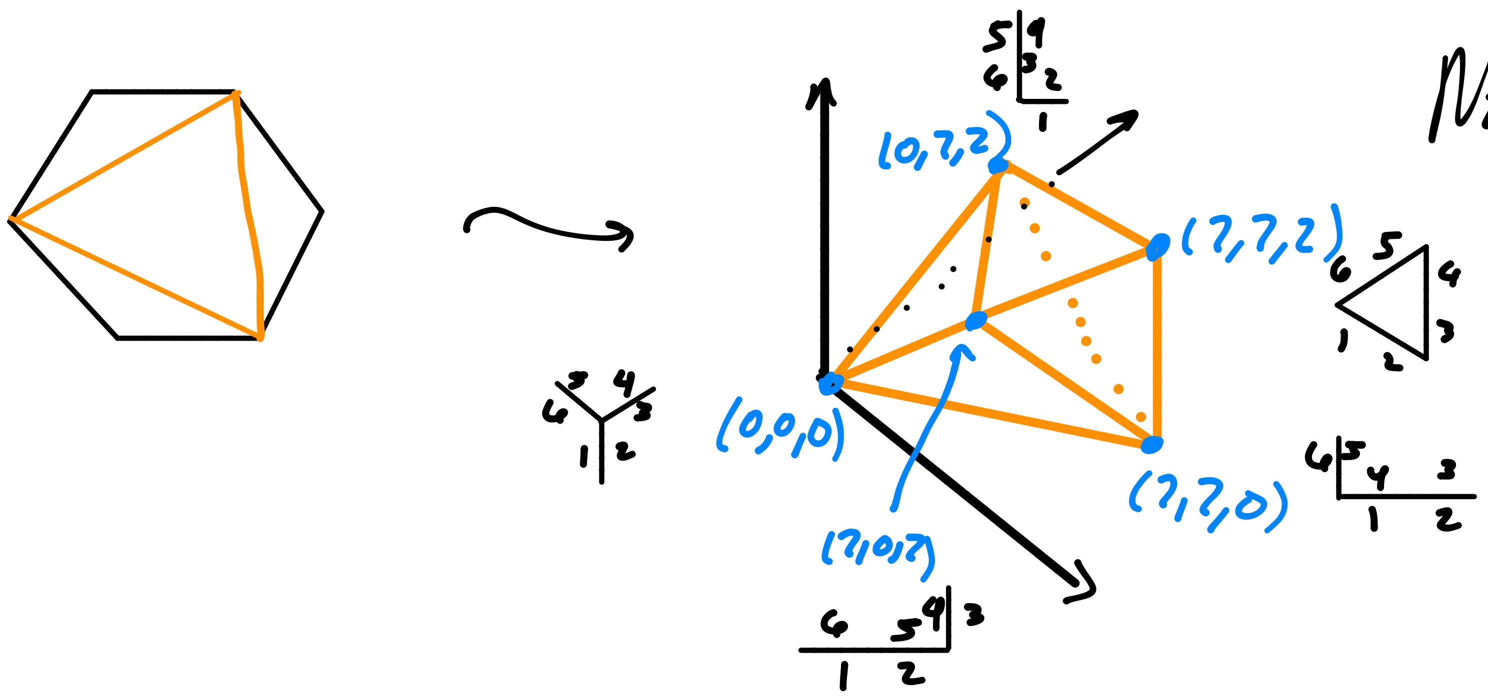
In fact, can get something which is toric (though not smooth) by identifying all such singular polygons w/ the same diagonal lengths & extending the torus action as the trivial action on these points; see Kamiyama–Yoshida.

Now, the restrictions on the chord lengths are just the triangle inequalities:



Notice that this is not Delzant: these two vertices are not smooth

$n=6$:



Notice that this is not Delzant: there are 4 edges incident to some vertices

Using the moment polytope & Duistermaat–Heckman, various people have computed the volume of Pol(n):

Thm (Takakura, Khoi, Kamiyama–Tezuka, Martin, Mandini, ...):

$$\mathrm{Vol}(\mathrm{Pol}(n)) = -\frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} (n-2k)^{n-3}$$

Q: Is there a nicer expression or interpretation of this expression?

One idea: it kind of looks like the second derivative of $\varphi_n(\ell) = \frac{2\ell}{\pi}\int_0^\infty \sin(\ell y)\,\mathrm{sinc}^n y\,dy$

In general, given any algorithm for sampling these polytopes, you get an algorithm for sampling $Pol(n)$ w.r.t. the Liouville measure

Thm (w/ Cantarella, Duplantier, Uehara): An explicit algorithm for sampling $Pol(n)$ in expected time $\Theta(n^{5/2})$.

Of course, it would be great to do better...

Also, a similar thing for planar polygons does not exist (planar polygons are, if you like, the GIT quotient $(\mathbb{RP}^1)^n /\!/ PSL(2,\mathbb{R})$, but there's no symplectic side of the story & thus no natural measure.