Math 476: Days 27-28

Prop: For each stable measure ν on S^2 $\exists$ $\gamma \in PSL(2,\mathbb{C})$ s.t. $B(\gamma_* \nu) = 0$. The element γ is unique up to postcomposition by $g \in SO(3) < PSL(2,\mathbb{C})$.

Why is the proposition true?

This is where we use the hyperbolic geometry: we want to define the hyperbolic center of mass of a measure on the sphere at infinity. Why is this the thing to do?

① If $C(\nu)$ is the hyperbolic center of mass of the stable measure ν on S^2 & $\gamma \in PSL(2,\mathbb{C})$, then

$$\gamma(C(\nu)) = C(\gamma_* \nu).$$

In other words, the hyperbolic center of mass is equivariant w.r.t. the $PSL(2,\mathbb{C})$ action on $\mathbb{H}^3 \cup S^2$. This is obvious since $PSL(2,\mathbb{C})$ acts by isometries on $\mathbb{H}^3$.

② $C(\nu) = \vec{0} \iff B(\nu) = \vec{0}$. We will see why this is true shortly.

Now, why are these 2 properties what we want?

Well, if $\nu = \frac{1}{2} \sum r_i \delta_{e_i}$ is the measure corresponding to a stable collection of weighted points, then it will turn out to be the case that $C(\nu)$ is in the interior of the unit ball. But then $\exists$ a unique (up to rotation) isometry γ of $\mathbb{H}^3$ sending $C(\nu)$ to $\vec{0}$. By ①, this means that

$$\vec{0} = C(\gamma_* \nu) = C\left(\tfrac{1}{2} \sum r_i \delta_{\gamma(e_i)}\right).$$

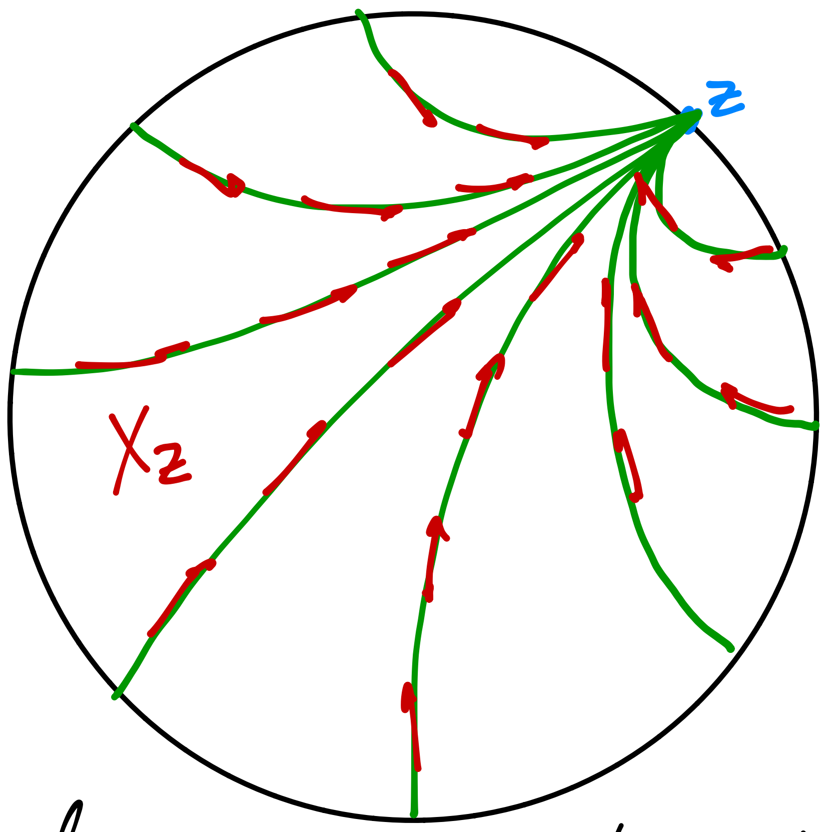
But then by ②, the Euclidean center of mass $B(\gamma_* \nu) = \sum \frac{r_i}{2} \gamma(e_i)$ is also at the origin, so $(\gamma(e_1), \ldots, \gamma(e_n))$, which is in the same $PSL(2,\mathbb{C})$-orbit as $(e_1, \ldots, e_n)$ & hence in the same equivalence class in $(\mathbb{P}^1)^n /\!/ PSL(2,\mathbb{C})$, corresponds to a closed polygon.

So now how should we define $C(\nu)$?

Well, at each point $p \in B^3 \cong \mathbb{H}^3$, we can define a vector $X_\nu(p)$ by

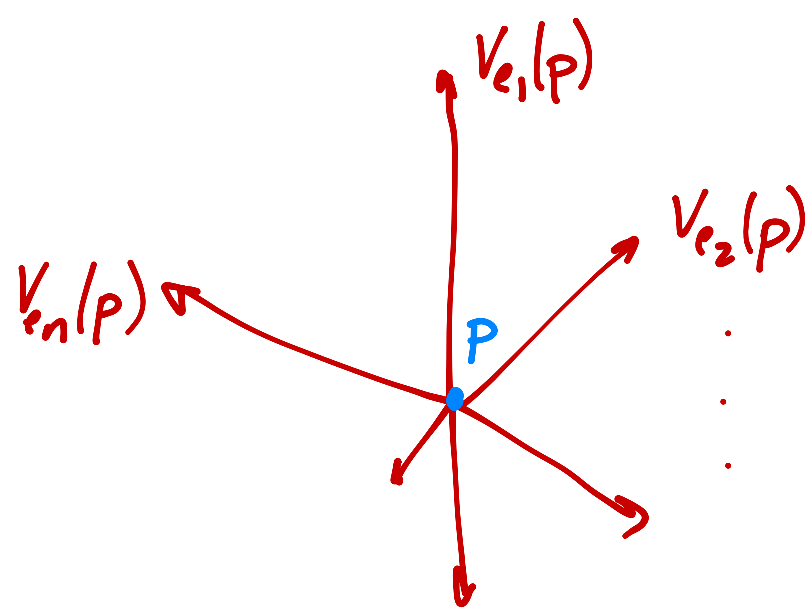
$$X_\nu(p) := \int_{S^2} V_z(p)\, d\nu(z)$$

where V_z is the unit vector field on $\mathbb{H}^3$ that points along the hyperbolic geodesic to $z \in S^2$:



In other words, $X_\nu(p)$ is the visual average of ν along all hyperbolic rays emanating from p.

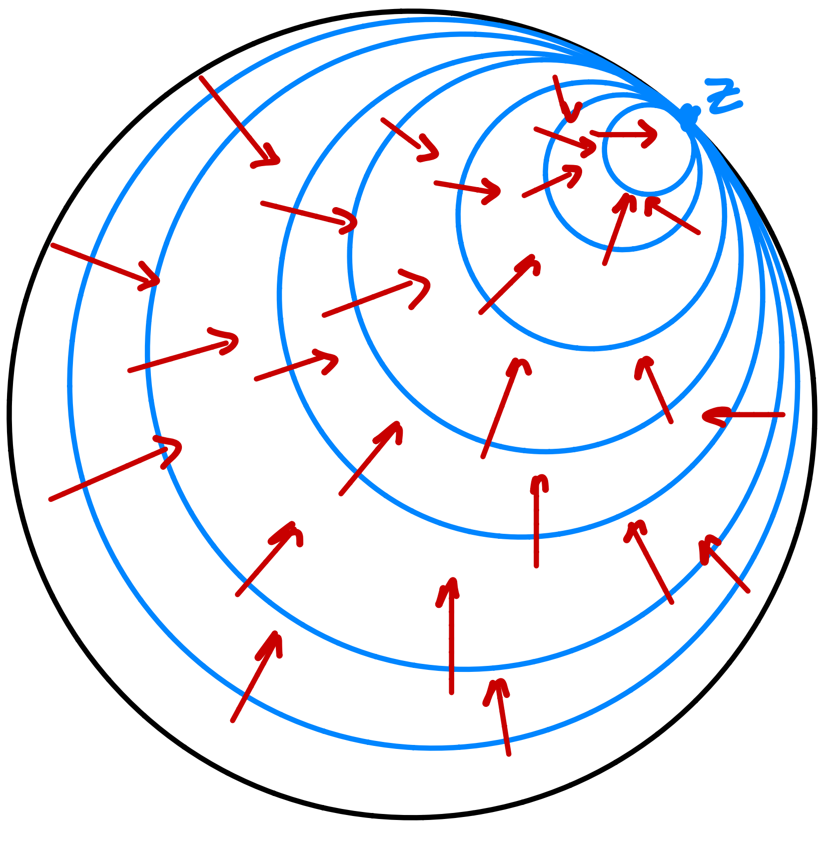
If ν is atomic, then $\int_{S^2} V_z(p)\, d\nu(z) = \sum \frac{r_i}{2} V_{e_i}(p)$:



But then the hyperbolic center of mass should be the (unique!) point where this is exactly balanced:

Def: The hyperbolic (or conformal) center of mass of the stable measure ν on S^2 is the unique zero of the vector field X_ν.

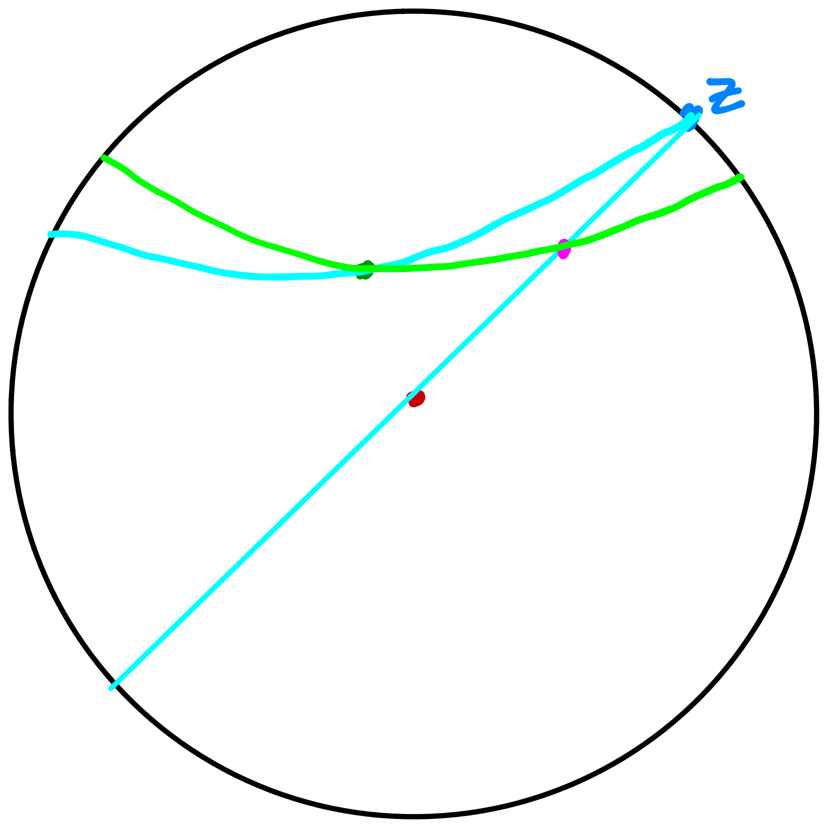
Equivalently, V_z is the gradient of a function $h_z : B^3 \to \mathbb{R}$ whose level sets are the horocycles centered at $z \in S^2$



of course, h_z is only defined up to a constant, so might as well choose $h_z(\vec{0})=0$.

In fact, $h_z(p)=\frac{1}{2}\log\left(\frac{1-|p|^2}{|z-p|^2}\right)$ & one can show that $-h_z(p)=-\frac{1}{2}\log\left(\frac{1-|p|^2}{|z-p|^2}\right)=\lim_{r\to 1^-}\left[d(p,rz)-d(0,rz)\right]$

where $d(x,y)$ is the hyperbolic distance from x to y.



But then $X_\nu=\nabla H_\nu$, where $H_\nu(p):=\int_{S^2} h_z(p)\,d\nu(z)$.

Then $C(\nu)$ is a critical point of H_ν, & existence & uniqueness follow because the restriction of

$-H_\nu$ to the hyperbolic geodesics is strictly convex.

Mathematica demo of closing by finding the conformal center of mass.