Math 676: Day 9

Symplectic Geometry

Def: A *symplectic manifold* is a manifold M together with a closed, nondegenerate 2-form $\omega \in \Omega^2(M)$.

Closed means $d\omega = 0$.

Nondegenerate means that, at each $p \in M$, the map $\varphi_{\omega_p} : T_pM \to T_pM^*$ given by $\varphi_{\omega_p}(v)(u) = \omega_p(v,u)$ is an isomorphism (equivalently, $\ker(\varphi_{\omega_p}) = \{0\}$, since we already know T_pM & T_pM^* are the same dimension)

In general, a skew-symmetric bilinear map $\omega : V \times V \to \mathbb{R}$ where V is a v.s. is called *symplectic* if $\varphi_\omega : V \to V^*$ is an isomorphism. The pair (V, ω) is called a *symplectic vector space*.

Ex: Consider $\mathbb{R}^{2n} = \mathbb{C}^n$ & define $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ by $\omega(\vec{u}, \vec{v}) = (i\vec{u}) \cdot \vec{v}$.

Then nondegeneracy follows from Riesz representation & the fact that mult. by i is an isomorphism.

This is the origin of the term "symplectic": "The name 'complex group' formerly proposed by me ... has become more and more embarrassing through collision with the word 'complex' in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective 'symplectic'." — Hermann Weyl, Classical Groups, p. 165.

↑ twining or plaiting together

A symplectic v.s. can always be equipped w/ a *symplectic basis* $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ s.t. $\omega(e_i, f_j) = \delta_{ij}$

& $\omega(e_i, e_j) = 0 = \omega(f_i, f_j)$.

w.r.t. this basis, ω has the form $\omega(u,v) = (-\, u \,-) \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$

(notice that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \pi/2 & \sin \pi/2 \\ -\sin \pi/2 & \cos \pi/2 \end{bmatrix}$ is the std. rep. of i as an elt. of $SO(2)$).

In particular, any symplectic v.s. must be *even-dimensional*.

Of course, this also implies a symplectic manifold must be even-dim'l, since the symplectic form makes each tangent space a symplectic vector space.

Ex: let $M=\mathbb{R}^{2n}$ w/ coords. $x_1,\dots,x_n,y_1,\dots,y_n$ & define

$$\omega := \sum_{i=1}^{n} dx_i\wedge dy_i$$

Then at each $p\in M$, $\varphi_{\omega_p}\left(\frac{\partial}{\partial x_i}\right)\left(\frac{\partial}{\partial y_i}\right) = \left(\sum dx_i\wedge dy_i\right)\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_i}\right)=1$ & likewise for $\varphi_{\omega_p}\left(\frac{\partial}{\partial y_i}\right)$, so φ_{ω_p} is

injective & so ω is nondegenerate.

then obviously $\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}\right\}$ is a symplectic basis for each tgt space

Also, $d\omega = d\sum dx_i\wedge dy_i = \sum\left[d(dx_i)\wedge dy_i - dx_i\wedge d(dy_i)\right]=0$, so ω is a symplectic form.

Notice that ω really is just the same "multiplication by i" form discussed above.

Ex: let $M=\mathbb{C}^n$ w/ coords $z_1,\dots,z_n$ & define

$$\omega := \frac{i}{2}\sum dz_k\wedge d\bar{z}_k$$

Then ω is symplectic... in fact, equals the ω from the previous ex. w/ $z_k = x_k+iy_k$.

Ex: $M=S^2$ w/ the form ω given by $\omega_p(u,v)=(u\times v)\cdot p$. $d\omega=0$ b/c all 3-forms on a 2-mfd are zero,

& we saw last time that ω is nondegenerate.

In fact, more generally, any oriented surface is a symplectic mfd, w/ the symplectic form given by the area form

(which exists since the surface is oriented)

Thm: If (M^{2n},ω) is a symplectic mfd, then $\omega^n := \underbrace{\omega\wedge\dots\wedge\omega}_{n}$ is a volume form.

pf: Exercise.

Therefore, a symplectic mfd has a canonical orientation & the volume form $\frac{\omega^n}{n!}$ is called the symplectic volume or Liouville

volume of (M,ω).

Ex: In our $\mathbb{R}^{2n}$ example, $\frac{\omega^n}{n!} = \frac{1}{n!}\left[\left(\sum dx_i\wedge dy_i\right)\wedge\dots\wedge\left(\sum dx_i\wedge dy_i\right)\right] = dx_1\wedge dy_1\wedge\dots\wedge dx_n\wedge dy_n$, so

the Liouville volume on $\mathbb{R}^{2n}$ induces Lebesgue measure.

Prop: Both ω & ω^n represent nontrivial deRham cohomology classes on the compact symplectic mfd (M,ω),

meaning they are closed (obvious) & not exact.

We see that, in particular, no sphere S^{2n} w/ $n>1$ can be symplectic, since such spheres have no nontrivial 2-dim'l cohomology class.

We now have a list of obstructions to a compact mfld M being symplectic: it must be even-dimensional, orientable, & have nontrivial cohomology in all even degrees.