Math 676: Day 8

We are about to dive into symplectic geometry, but first I want to give some sense of why we will care about symplectic geometry at all.

The basic issue is that we want to sample points uniformly from our polygon space

$$\mathrm{Pol}(n) = (\underbrace{S^2 \times \ldots \times S^2}_{n}) /\!\!/_{\vec{0}} SO(3)$$

The nice things that symplectic geometry will give us are:

① Access to the correct measure on this space

& ② Good coordinates to sample from that measure

Sampling Issues

In order to sample (randomly) from a set/space X, we need a probability measure on X.

In probabilistic language, we will have a sample space $(X, \mathcal{B}, P)$, where $\mathcal{B}$ is a σ-algebra & P is a measure on $(X,\mathcal{B})$ s.t. $P(X) = 1$.

Remember that a measure μ on a measurable space $(X, \mathcal{A})$ (where X is a set & $\mathcal{A}$ is a σ-alg. on X) is a map

$\mu : X \to [0, +\infty]$ s.t. ① $\mu(\emptyset) = 0$

② For $E_1, E_2, \ldots \in \mathcal{A}$ pairwise disjoint, $\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) + \mu(E_2) + \ldots$

(A σ-algebra $\mathcal{A}$ on X is a nonempty subset of $\mathcal{P}(X)$ that is closed under countable unions & complementation)

Idea: The σ-alg. tells you what sets you can measure; it turns out to lead to contradictions if you try to measure every subset of X. For most σ-algebras, all the sets you'd care to measure are there.

Ex: The Borel σ-algebra on a topological space X is generated from the open sets of X by iteratively taking all possible countable unions & complements

formally, this requires some transfinite induction, but the point is that all reasonable sets you'd be likely to think of are Borel sets and, e.g., all (compactly supported) continuous functions on X are measurable.

For any sort of measure we might encounter in this class, any measurable set E can be written as

① $E = U \setminus N$, where U is a countable intersection of open sets (a G_δ set) & N has measure 0

and ② $E = K \cup N'$ where K is a countable union of closed sets (an F_σ set) & N' has measure 0.

Now, back to probability...

An *event* is just a measurable set E & the probability of the event is just the measure $P(E)$ of the set.

So notice that, from the above discussion, for probabilistic purposes it's enough to know the probabilities of G_δ or F_σ sets/events.

Now, to *sample* from a sample space $(X, \mathcal{B}, P)$ (when the measurable space $(X, \mathcal{B})$ is implied/known, this is often said as "to sample from a distribution P on X")

means to select a point $x \in X$ s.t. $P(x \in E) = P(E)$ $\forall$ $E \in \mathcal{B}$.

So now, let's think about what this means on manifolds. First of all, let's assume our mfld M is compact so that we don't have to worry about integrals converging.

Now, think about if $M \subseteq \mathbb{R}^n$ is a compact (or even just bounded) domain. Then we know perfectly well what the measure on M should be: clearly, if $E = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$, then the measure of E should be $(b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$. But then we just take the measurable sets to be those generated by such boxes & extend the measure in the (more or less) obvious way. This is called the *Lebesgue measure* λ on M.

If we want a probability measure P, we just normalize: $P(E) = \frac{\lambda(E)}{\lambda(M)}$ for $E \subseteq M$ measurable.

Okay, so you can do the same sort of thing on manifolds by using the above technology on coordinate charts & patching together w/ a partition of unity, but there's a cleaner way (which is really equivalent, but sweeps the mess under the rug):

Lebesgue integral

If $M \subseteq \mathbb{R}^n$ & $E \subseteq M$ is Borel, $\lambda(E) = \int_E d\lambda(x_1, \ldots, x_n) = \int_E dx_1 \wedge \cdots \wedge dx_n$

The point is that the standard *volume form* $dx_1 \wedge \cdots \wedge dx_n$ tells you how to measure volumes (in the Lebesgue sense) of (infinitesimal) boxes, & the whole point of the definition of the Lebesgue integral is that it makes this equality work.

Def: A *volume form* γ on an n-dim'l mfld M is an n-form $\gamma \in \Omega^n(M)$ s.t. γ never vanishes, meaning at each $p \in M$, $\exists$ $v_1, \ldots, v_n \in T_pM$ s.t. $\gamma(v_1, \ldots, v_n) \neq 0$.

In $\mathbb{R}^n$, $dx_1 \wedge \dots \wedge dx_n$ is obviously a volume form, since $dx_1 \wedge \dots \wedge dx_n\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) = 1$ at all $p \in \mathbb{R}^n$.

But there are lots of volume forms on $\mathbb{R}^n$ & they all induce measures on $M \subseteq \mathbb{R}^n$.

So, in general, any volume form γ on a
n-fold $M \subseteq \mathbb{R}^n$ will induce a probability measure P_γ on M given by

$$P_\gamma(E) = \frac{\int_E \gamma}{\int_M \gamma} \quad \text{for all Borel sets } E \subseteq M.$$

But now the exact same definition gives a probability measure on any compact orientable mfld M.

Why orientable? Well, volume forms only exist on orientable mflds: the volume form γ is a section of

$$\begin{array}{c} \bigcup_{p \in M} \Lambda^n(T_pM^*) \\ \downarrow \\ M \end{array}$$

which misses the zero section completely. Since $\Lambda^n(T_pM^*) \cong \mathbb{R}$, this means a consistent choice of component of

$\Lambda^n(T_pM^*) \setminus \{0\} \cong \mathbb{R} \setminus \{0\}$, which means a consistent choice of orientation of T_pM^*, which is what an orientation on a

mfld is.

Ex: Consider S^2 as the unit sphere in $\mathbb{R}^3$, & define $\omega_{S^2} \in \Omega^2(S^2)$ by

$\omega_p(\vec{u}, \vec{v}) = (\vec{u} \times \vec{v}) \cdot \vec{p}$, for $p \in S^2$, $\vec{u}, \vec{v} \in T_pS^2$, realized as vectors in $\mathbb{R}^3$ perpendicular to p.

Ex: Again on S^2, we can use cylindrical coords (r, θ, z) (at least away from the poles), where, for example, $r = \sqrt{1-z^2}$, so (θ, z) give almost global coords on S^2. Then $\omega_{S^2} = d\theta \wedge dz$. (Exercise)

Ex: If $M_1, \dots, M_n$ are mflds w/ volume forms $\omega_1, \dots, \omega_n$, then $\omega_1 \wedge \dots \wedge \omega_n$ is a volume form on $M_1 \times \dots \times M_n$, & the induced measure agrees w/ the product measure.