Math 476: Day 6

Ex: On $\mathbb{R}^2$, we have global coordinate functions x & y; formally, $x(a,b)=a$, $y(a,b)=b$. Then $dx, dy \in \Omega^1(\mathbb{R}^2)$ & $\{dx, dy\}$ gives a basis for each $(T_p\mathbb{R}^2)^*$, so every $\omega \in \Omega^1(\mathbb{R}^2)$ is of the form $\omega = f\,dx + g\,dy$.

Of course, $\omega_p\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = (f\,dx+g\,dy)_p\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right) = f(p)\,a(p) + g(p)\,b(p)$

Also, if $\omega = f\,dx + g\,dy$, $\eta = h\,dx + m\,dy$, then $\omega\wedge\eta = (f\,dx+g\,dy)\wedge(h\,dx+m\,dy) = fm\,dx\wedge dy + gh\,dy\wedge dx$

$$= (fm - gh)\,dx\wedge dy$$

Now, $\omega = f\,dx + g\,dy$ is dual to the v.f. $V = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$.

We can compute the planar curl of V: $\nabla\times V = \nabla\times\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$

The corresponding operation on $\Omega^1(\mathbb{R}^2)$ is the exterior derivative d:

$$d\omega = d(f\,dx+g\,dy) = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)\wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)\wedge dy = \frac{\partial f}{\partial y}dy\wedge dx + \frac{\partial g}{\partial x}dx\wedge dy$$
$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy$$

Now, there's a much nicer way to write Green's Thm in the plane using Stokes' Thm for Differential Forms:

Stokes' Thm: If M^n is a mfld w/ bdry & $\omega\in\Omega^{n-1}(M)$, then

$$\int_M d\omega = \int_{\partial M}\omega$$

So in this case, if $M\subseteq\mathbb{R}^2$ is a domain & $\omega = f\,dx+g\,dy$, then

$$\int_{\partial M} f\,dx + g\,dy = \int_M\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy,$$

which should be familiar

Ex: On $\mathbb{R}$, all 0-forms are just functions on $\mathbb{R}$, so Stokes' Thm says

$$\int_{[a,b]}\frac{\partial f}{\partial x}dx = \int_{\{b\}} f - \int_{\{a\}} f = f(b) - f(a).$$

Ex: On $\mathbb{R}^3$, $\omega\in\Omega^1(\mathbb{R}^3)$ is of the form $\omega = f\,dx + g\,dy + h\,dz$. If $\eta = (a\,dx + b\,dy + c\,dz)$, then

$$\omega\wedge\eta = fb\,dx\wedge dy + fc\,dx\wedge dz + ga\,dy\wedge dx + gc\,dy\wedge dz + ha\,dz\wedge dx + hb\,dz\wedge dy$$
$$= (gc - hb)\,dy\wedge dz + (ha - fc)\,dz\wedge dx + (fb - ga)\,dx\wedge dy.$$

Compare to $\left(f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}+h\frac{\partial}{\partial z}\right)\times\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\frac{\partial}{\partial z}\right)=(gc-hb)\frac{\partial}{\partial x}-(fc-ha)\frac{\partial}{\partial y}+(fb-ga)\frac{\partial}{\partial z}$.

Also, $d\omega=d(f\,dx+g\,dy+h\,dz)=\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz\right)\wedge dx+\left(\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy+\frac{\partial g}{\partial z}dz\right)\wedge dy+\left(\frac{\partial h}{\partial x}dx+\frac{\partial h}{\partial y}dy+\frac{\partial h}{\partial z}dz\right)\wedge dz$

$$=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dy\wedge dz+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dz\wedge dx+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dx\wedge dy$$

Compare to $\nabla\times\left(f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}+h\frac{\partial}{\partial z}\right)=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\frac{\partial}{\partial x}-\left(\frac{\partial h}{\partial x}-\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial y}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\frac{\partial}{\partial z}$

Now, 2-forms look like $\alpha=f\,dy\wedge dz+g\,dz\wedge dx+h\,dx\wedge dy$, so Stokes' Thm says that, if $M\subseteq\mathbb{R}^3$ is a domain,

$$\int_M\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)dx\wedge dy\wedge dz=\int_{\partial M}f\,dy\wedge dz+g\,dz\wedge dx+h\,dx\wedge dy$$

Compare to the usual Divergence Thm

$$\int_M\nabla\cdot\left(f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}+h\frac{\partial}{\partial z}\right)dV=\int_{\partial M}\left(f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}+h\frac{\partial}{\partial z}\right)\cdot\vec{n}\,dS$$

$$\parallel$$

$$\int_M\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)dV$$

In general, on $\mathbb{R}^n$, there is no curl operator, but the exterior derivative of a 1-form is the right generalization.

Also, the forms $dx\wedge dy$, $dx\wedge dy\wedge dz$, etc. are examples of volume forms: a volume form on M^n is a nowhere-vanishing element of $\Omega^n(M)$.

Another example: The 2-form ω on S^2 given by $\omega_p(\vec{u},\vec{v})=(\vec{u}\times\vec{v})\cdot\vec{p}$, which is obviously nowhere-vanishing since $(\vec{u}\times\vec{v})\cdot\vec{p}\neq 0$ so long as $\vec{u}$ & $\vec{v}$ are lin. indept elts of T_pS^2.

In coords, $\omega_p(\vec{u},\vec{v})=(\vec{u}\times\vec{v})\cdot\vec{p}=(u_2v_3-u_3v_2)p_1+(u_3v_1-u_1v_3)p_2+(u_1v_2-u_2v_1)p_3$,

so $\omega=x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy$.

Ok, so in local coords the exterior derivative on an arbitrary mfld is the same thing. In a coord. chart (U,φ) on M near p, have local coord. functions $x_1,\dots,x_n$, where $x_i(\varphi(a_1,\dots,a_n))=a_i$, where $(a_1,\dots,a_n)\in U$.

Then for all k-elt sets $I=\{i_1\le\dots\le i_k\}\subseteq\{1,\dots,n\}$, the k-forms $dx_I:=dx_{i_1}\wedge\dots\wedge dx_{i_k}$ form a basis for T_pM, so can locally write any elt. of $\Omega^k(M)$ as $\omega=\sum_I f_I\,dx_I$, &

$$(d\omega)_p=\sum(df_I)\wedge dx_I.$$

Here's the coordinate-free definition:

Def: The exterior derivative $d : \Omega^{\cdot}(M) \to \Omega^{\cdot}(M)$ is the unique anti-derivation of degree $+1$ s.t.:

① $\forall f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f.

② $d \circ d = 0$.

In general, given a graded algebra A, a derivation of A is a map $D : A \to A$ s.t. $D(a*b) = D(a)*b + a*D(b)$

an anti-derivation is a map $D : A \to A$ s.t. $D(a*b) = D(a)*b + (-1)^p a*D(b)$ when $a \in A^p$

An (anti-)derivation has degree k if $D : A^i \to A^{i+k}$.

The exterior derivative turns $\Omega^{\cdot}(M)$ into a (co-)chain complex

$$0 \xrightarrow{d_{-1}} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} \Omega^{n-1}(M) \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0$$

The (co)homology of this complex is called the de Rham cohomology of M; i.e.,

$$H^k_{dR} = \frac{\ker d_k}{\operatorname{im} d_{k-1}}$$

w/ cup product induced by the wedge product.

Thm: $H^*_{dR}(M) \cong H^*(M; \mathbb{R})$ as graded rings

↑ singular cohomology