Math 676: Day 5

Remember, we "defined" k-forms on a mfd M to be smooth, alternating, multilinear maps $\underbrace{\mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)}_{k}\to\mathbb{R}$.

Ex: A **Riemannian metric** on a mfd M is g which at each $p\in M$ determines an inner product g_p on T_pM s.t. g_p varies smoothly with p (more formally, g is a smooth section of the **(0,2)-tensor bundle** on M s.t.

$g(p) =: g_p$ is an inner product)

Let M be a mfd & $X\in\mathfrak{X}(M)$ a vector field. Assume M has a Riemannian metric g.

Then X corresponds to a unique 1-form α defined by $\alpha_p(Y)=g_p(X,Y)$ (i.e. $\alpha_p = g_p(X,\cdot)$)

Then $\alpha:\mathfrak{X}(M)\to C^\infty(M)$ is smooth since the Riemannian metric is.

It's linear since g is. It's trivially alternating

OTOH, any 1-form β is just a linear functional on each T_pM, so $\beta_p\in(T_pM)^*$ $\forall\, p\in M$. But then

$\exists\, u_p\in T_pM$ s.t. $\beta_p(v)=g_p(u_p,v)$ $\forall\, v\in T_pM$ (this is the **Riesz Representation Thm**). Then the smoothness of β allows us to define a smooth vector field U by $U_p=u_p$.

In other words, g determines an isomorphism $\mathfrak{X}(M)\to\Omega^1(M)$ by $X\mapsto g(X,\cdot)$.

Also, 0-forms are clearly just C^∞ functions on M, so $\Omega^0(M)=C^\infty(M)$.

Ex: An n-form on M^n is an alternating multilinear map $\omega:(\mathfrak{X}(M))^n\to C^\infty(M)$. At each $p\in M$, $T_pM\cong\mathbb{R}^n$, so ω_p is an alternating multilinear map $\underbrace{\mathbb{R}^n\times\dots\times\mathbb{R}^n}_{n}\to\mathbb{R}$. Up to a scalar, the unique such thing is the determinant (to make this rigorous, need a Riemannian metric), so each ω_p is a scalar multiple of the determinant, meaning that $\omega=f\,d\mathrm{vol}_M$, where $d\mathrm{vol}_M$ is the name we give to the n-form which is just the determinant on each T_pM. Therefore $\Omega^n(M)\simeq C^\infty(M)$

Tensor Algebras & Exterior Algebras

To define differential forms properly (& to turn forms into a graded algebra), we need to do some (linear) algebra.

Given vector spaces V & W, define the **tensor product** $V\otimes W$ to be the vector space of elements $v\otimes w$ for $v\in V, w\in W$, s.t.

① $(v_1+v_2)\otimes w = v_1\otimes w + v_2\otimes w$

② $v\otimes(w_1+w_2) = v\otimes w_1 + v\otimes w_2$

③ $a(v\otimes w) = (av)\otimes w = v\otimes(aw)$

Ex: If $V=\mathbb{R}^m$, $W=\mathbb{R}^n$, then $\vec{v}\otimes\vec{w}$ corresponds to the $m\times n$ matrix $\vec{v}\,\vec{w}^T = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}[w_1 \cdots w_n]$

Thm (Universal Property): If $\varphi: V\times W \to V\otimes W$ is the map given by $(v,w)\mapsto v\otimes w$, & $F: V\times W\to U$ is bilinear, then $\exists!$ linear map $\tilde{F}: V\otimes W\to U$ making the diagram commute:

$$\begin{tikzcd} V\times W \arrow[r, "F"] \arrow[d, "\varphi"'] & U \\ V\otimes W \arrow[ur, dashed, "\exists!\,\tilde{F}"'] & \end{tikzcd}$$

Moreover, this uniquely characterizes $V\otimes W$: any v.s. satisfying this property is isomorphic to $V\otimes W$.

The point is that tensor products are a way of turning bilinear (or, more generally, multilinear) maps into linear maps.

Prop: If $e_1,\dots,e_m$ is a basis for V & $f_1,\dots,f_n$ is a basis for W, then $\{e_i\otimes f_j\}_{i,j}$ is a basis for $V\otimes W$.

In particular, $\dim V\otimes W = mn$.

Differential forms are almost tensor fields, but we need to take an appropriate quotient to incorporate the alternating condition. Specifically, just as the tensor product is the way to turn multilinear maps into linear maps, the exterior product is the way to turn alternating, multilinear maps into linear maps.

Def: Let $C(V)=\bigoplus_k V_k^0$ be the algebra of $(\cdot,0)$ tensors on a v.s. V & let $I(V)$ be the two-sided ideal generated by elements of the form $v\otimes v$ for $v\in V$. Then the exterior algebra of V is the quotient

$$\Lambda(V) := C(V)/I(V).$$

This is a graded algebra w/ product denoted by $\wedge$. Of course, the product is just that induced by $\otimes$: the residue class of $v_1\otimes\dots\otimes v_k$ is denoted $v_1\wedge v_2\wedge\dots\wedge v_k$.

The grading is given by the degree of the wedge product, since

$$\Lambda(V)=\bigoplus_{k\geq 0}\Lambda^k(V)$$

where $\Lambda^k(V)=V_k^0/I_k$ & $I_k = I(V)\cap V_k^0$.

Prop: If $a \in \Lambda^k(V)$, $b \in \Lambda^\ell(V)$, then $b \wedge a = (-1)^{k\ell} a \wedge b$.

Proof: Exercise. $\boxtimes$

Cor: $v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)} = (-1)^{\operatorname{sgn}(\sigma)} v_1 \wedge \dots \wedge v_k$ for $\sigma \in S_k$.

Prop: If $\{e_1, \dots, e_n\}$ is a basis for V, then $\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \le i_1 < i_2 < \dots < i_k \le n\}$ is a basis for $\Lambda^k(V)$. In particular,

$$\dim \Lambda^k(V) = \binom{n}{k} \text{ for } 0 \le k \le n \text{ \& zero otherwise, so } \dim \Lambda(V) = 2^n.$$

Proof: Exercise. $\boxtimes$

This is kind of awesome b/c finite-dim'l vector spaces are always nicer than infinite-dim'l (note the tensor algebra is ∞-dim'l).

As you would expect, $\Lambda(V)$ also satisfies a universal property:

Thm (Universal Property): If $\varphi : \underbrace{V \times \dots \times V}_{k} \to \Lambda^k(V)$ is given by $(v_1, \dots, v_k) \mapsto v_1 \wedge \dots \wedge v_k$ & if $F : V \times \dots \times V \to U$ is an alternating multilinear map, then $\exists!$ linear map $\tilde{F} : \Lambda^k(V) \to U$ s.t. the diagram commutes:

$$\begin{tikzcd}
V \times \dots \times V \arrow[r, "F"] \arrow[d, "\varphi"'] & U \\
\Lambda^k(V) \arrow[ur, dotted, "\exists! \tilde{F}"'] &
\end{tikzcd}$$

Again, this property uniquely defines $\Lambda^k(V)$ up to isomorphism.

In particular, when $U = \mathbb{R}$ (or whatever the base field is), we have that $\Lambda^k(V)^* \cong A_k(V)$, the space of alternating, multilinear maps on V^k.

Now, we can define the exterior k bundle on M:

$$\Lambda^k(M) := \bigcup_{p \in M} \Lambda^k((T_pM)^*)$$

And the exterior algebra bundle over M:

$$\Lambda(M) := \bigcup_{p \in M} \Lambda((T_pM)^*)$$

Then a differential k-form on M is a smooth section of $\Lambda^k(M)$; i.e., a smooth map $\omega : M \to \Lambda^k(M)$ s.t. $\omega(p) \in \Lambda^k((T_pM)^*)$.

And now it should start to become clear that this is just another way of saying a k-form is an alternating, multilinear map $\mathfrak{X}(M)^k \to C^\infty(M)$
(provided one shows $\Lambda^k(V)^* \cong \Lambda^k(V^*)$, which it is)