Math 676: Day 4

Vector Fields & Brackets

Ex: Consider $SO(n)$. Then an element of $T_I SO(n)$ is a tangent vector; i.e., the tangent to a curve through I. Let

$\alpha: (-\varepsilon, \varepsilon) \to SO(n)$ s.t. $\alpha(0) = I$. Then $\alpha(t)$ is an orthogonal matrix $\forall\, t \in (-\varepsilon, \varepsilon)$, so $\alpha(t)\alpha(t)^T = I$.

Differentiate: $\alpha'(t)\alpha(t)^T + \alpha(t)\alpha'(t)^T = 0$. At $t=0$, this reduces to $\alpha'(0) + \alpha'(0)^T = 0$, so $\alpha'(0) = -\alpha'(0)^T$, so $\alpha'(0)$ is skew-symmetric. The converse is also true, so we can identify $T_I SO(n)$ w/ the set of skew-symmetric $n \times n$ matrices.

Also, note that if $\Delta \in T_I SO(n)$, then $e^{\Delta} \in SO(n)$ b/c $(e^{\Delta})^T = e^{(\Delta^T)} = e^{-\Delta} = (e^{\Delta})^{-1}$ & $\det(e^{\Delta}) = e^{\operatorname{tr}\Delta} = e^0 = 1$. You can prove that this map is actually surjective as well.

Now, if $Q \in SO(n)$ & we define $l_Q : SO(n) \to SO(n)$ by $l_Q(A) = QA$, then

$$(dl_Q)_I \Delta = Q\Delta \qquad \text{(check this if you don't believe it!)}$$

Since $(dl_Q)_I : T_I SO(n) \to T_I SO(n)$ is full rank it's surjective, so $T_Q SO(n)$ consists of matrices of the form $Q\Delta$ for Δ skew-symmetric.

Therefore, a vector field X on $SO(n)$ is given at each pt. $Q \in SO(n)$ by $X_Q = Q\Delta_Q$.

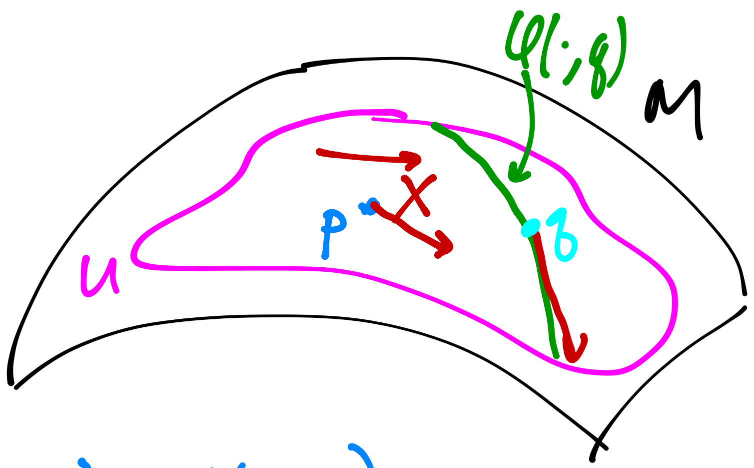
If $\Delta_Q = \Delta_I$ $\forall\, Q \in SO(n)$ is called *left-invariant*, & the space of left-inv't vf's on $SO(n)$ can then be identified w/ $T_I SO(n)$.

Now, the Lie bracket on left-inv't vf's is especially nice: if X & Y are left-inv't vf's w/ $X_I = \Delta_X$ & $Y_I = \Delta_Y$, then $[X,Y]_I = \Delta_1\Delta_2 - \Delta_2\Delta_1$ & $[X,Y]_Q = Q(\Delta_1\Delta_2 - \Delta_2\Delta_1)$, so the Lie bracket becomes a binary operation on $T_I SO(n)$. This makes $T_I SO(n)$ a **Lie algebra** denoted $\mathfrak{so}(n)$.

Def: Let X be a v.f. on M. A curve $\alpha: (a,b) \to M$ is called an *integral curve* (or *trajectory*) for X if $\alpha'(t) = X_{\alpha(t)}$ $\forall\, t \in (a,b)$.

Prop: Let X be a v.f. on M & let $p \in M$. Then $\exists$ nbhd U of p, $\delta > 0$, & map $\varphi: (-\delta, \delta) \times U \to M$ s.t. $t \mapsto \varphi(t,q)$ is the unique smooth curve satisfying $\frac{\partial \varphi}{\partial t} = X_{\varphi(t,q)}$ & $\varphi(0,q) = q$.

Proof: Since M is locally diffeo. to $\mathbb{R}^n$, this follows from the fundamental theorem on existence & uniqueness of solutions to ODEs. ∎

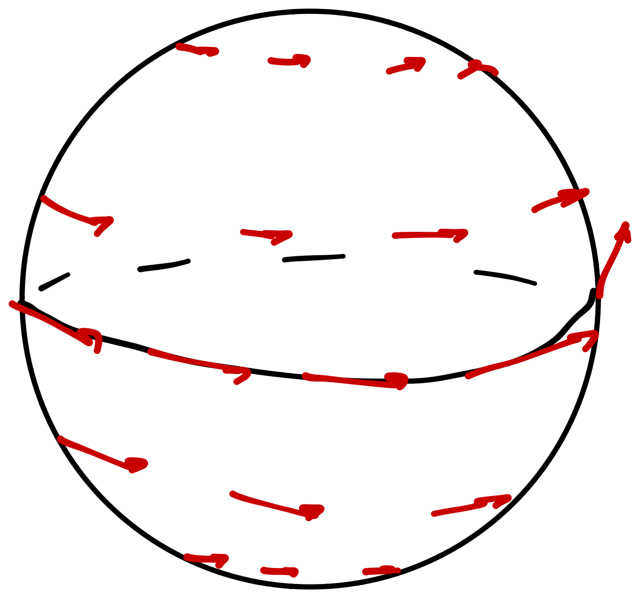


often denote $\varphi_t(q) = \varphi(t, q)$ & call it the local flow of X.

Ex: $M = S^2$, $X = \frac{\partial}{\partial\theta}$ (in cylindrical coords)

Then (in the complement of the north & south poles) $\varphi_t(r, \theta, z) = (r, \theta + t, z)$

This is an example of a complete v.f. since the local flow exists for all time at all points.



Now, we can interpret the Lie bracket $[X, Y]$ as a sort of derivative of Y along the trajectories of X.

The basic issue is: how do you form the difference quotient $\frac{Y_{\varphi_t(p)} - Y_p}{t}$? After all, $Y_{\varphi_t(p)} \in T_{\varphi_t(p)}M$, but

$Y_p \in T_pM$, & you can't subtract elements of completely different vector spaces.

Of course, the vector spaces are isomorphic, so you could move one to the other by some isomorphism, but which one?

The answer, of course, is $d\varphi_t$:

Prop: Let X & Y be smooth vector fields on M, let $p \in M$ & let φ_t be the local flow of X in a nbd of p. Then

$$[X, Y]_p = \lim_{t \to 0} \frac{Y_{\varphi_t(p)} - d\varphi_t(Y_p)}{t}$$

Differential Forms

"Def": A (smooth) differential k-form on a mfd M is a C^∞, alternating, multilinear map

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k \text{ copies}} \to C^\infty(M)$$

where $\mathfrak{X}(M)$ is the $C^\infty(M)$ module of smooth vector fields on M.

(i.e., $\omega(X_1, .., X_{i-1}, fX + gY, X_{i+1}, .., X_k) = f\,\omega(X_1, .., X_{i-1}, X, X_{i+1}, .., X_k) + g\,\omega(X_1, .., X_{i-1}, Y, X_{i+1}, .., X_k)$

& $\omega(X_1, \ldots, X, .., X, .., X_k) = 0$)

The vector space of k-forms is denoted $\Omega^k(M)$.

This def. is kind of vague & useless, so let's see a concrete ex.

Ex: Let $f: M \to \mathbb{R}$ be smooth. Then the differential df is a smooth 1-form:

Clearly $df: \mathfrak{X}(M) \to C^\infty(M)$ & it's smooth since f is.

It's linear & it's trivially alternating, so it's a 1-form according to our vague defn.