Math 676 Day 38: Open Problems

This is a list of open problems that I think are interesting and would love to discuss further (or to hear about your solutions of). Some of these problems should be fairly easy, others may very well be incredibly difficult. In all cases, even just some numerical explorations would probably lead to interesting things.

What does the algebraic geometry get you? We have given an interpretation of the space $\text{Pol}(n;\vec{1})$ of equilateral polygons in $\mathbb{R}^3$ as the GIT quotient $(\mathbb{CP}^1)^n /\!\!/ PGL(2,\mathbb{C})$. Of course, this is an interesting projective variety, birationally equivalent to $\overline{\mathcal{M}}_{0,n}$ and appearing as a limit of Hassett spaces, but the question is: does this have any usefulness in the study of random polygons and polymers? So far I don't know of any applications to random polygons that bypass the symplectic side of the story. For example, would knowledge of the intersection theory (i.e., the Chow ring) on this variety tell you anything useful about polygons?

Also, my understanding is the $\overline{\mathcal{M}}_{0,n}$ has a natural Kähler class called the *Weil–Peterson class*. Obviously, this Kähler class in conjunction with the complex structure induces a unique Riemannian metric. Is this metric on $\overline{\mathcal{M}}_{0,n}$ related to the metric on $\text{Pol}(n;\vec{1})$? If so, how? Of course, a Riemannian metric induces a volume form and hence a measure, so it is sensible to talk about sampling random points on $\overline{\mathcal{M}}_{0,n}$ with respect to this measure. Is this possible computationally? And would it be remotely interesting?

What can we say about quadrilaterals? Perhaps as a first step to addressing the previous problem, can we use our (algebraic-geometric) understanding of $\text{Pol}(4;\vec{1})$ to better understand quadrilaterals? We know that $\text{Pol}(4;\vec{1})$ is a $\mathbb{CP}^1$ (with three singular points), so coordinates are kind of obvious. Earlier in the semester I used an analogous interpretation of planar quadrilaterals as an $\mathbb{RP}^1$ to see that, with $x \in \mathbb{R}$ giving a coordinate on $\mathbb{RP}^1$, the convex polygons corresponded to $x < 0$ and the two different types of non-convex quadrilaterals to the intervals $(0,1)$ and $(1,\infty)$. Is there an analogous story for space quadrilaterals and some nice decomposition of $\mathbb{C}\backslash\{0,1\}$ into regions that correspond to different classes of quadrilaterals in $\mathbb{R}^3$?

A planar theory? Hopefully this course has conveyed the fact that there are two nice ways of understanding the geometry of the space $\text{Pol}(n;\vec{1})$ of equilateral polygons in $\mathbb{R}^3$: as a symplectic manifold (sometimes with isolated singularities) with an (almost) toric structure which arises as the symplectic reduction of $(S^2)^n$ by the diagonal $SO(3)$ action, and as a projective variety as described above in *What does the algebraic geometry get you?* For the purposes of sampling, numerical integration, etc., the symplectic version of the story seems to be more useful, since symplectic manifolds come equipped with canonical volume forms.

The algebraic version of this story still works for equilateral polygons in $\mathbb{R}^2$: this space can be interpreted as the GIT quotient of $(\mathbb{RP}^1)^n$ by the diagonal $PGL(2,\mathbb{R})$ action, which is birationally equivalent to the real moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ that has been studied by Satyan Devadoss and others (see, e.g., [1]). However, it is unclear to me how to get my hands on the natural measure on this space (i.e., thought of as a quotient of the submanifold of $(S^1)^n$ of those elements whose vector sum is $\vec{0}$; since $(S^1)^n$ has a natural Riemannian metric, this submanifold inherits a Riemannian metric which induces a metric on the quotient) by way of (real) algebraic geometry. So the question is: can one understand the measure (even if not the Riemannian metric) using the algebraic geometry? And, if not, is there a parallel geometric story which plays the role in this world of the symplectic reduction approach to polygons in space?

What do stable polygons tell us about $\overline{\mathcal{M}}_{0,n}$? In the last couple of weeks, I tried to give an overview of Yi Hu's theory of *stable polygons* [2], which give a space $\overline{\text{Pol}}(n;\vec{r})$ which is biholomorphic to $\overline{\mathcal{M}}_{0,n}$. His main motivation in writing the paper seems to have been to give tools for understanding the Kähler cone (cone of possible symplectic forms) of $\overline{\mathcal{M}}_{0,n}$, and he gives a conjecture along these lines. So far as I can tell, nobody has really done anything with these ideas since the paper came out in 1999. This might just indicate that it's not a useful interpretation, but I wonder: is there anything interesting one can say about $\overline{\mathcal{M}}_{0,n}$ by using the stable polygons interpretation?

What about $\overline{\mathcal{M}}_{g,n}$? Equilateral polygons in $\mathbb{R}^3$ have a reasonably natural connection to $\overline{\mathcal{M}}_{0,n}$, which means that $\overline{\mathcal{M}}_{0,n}$ is an object of interest in stochastic geometry and statistical physics. What about $\overline{\mathcal{M}}_{g,n}$ for $g \geq 1$? Is there some physical interpretation of points on, say $\overline{\mathcal{M}}_{1,n}$? In a paper I have tried to read many times but never really understood, Thurston [5] gives some (implicit) connection between cone metrics on the sphere with fixed cone angle (or, if you like, the space of convex polyhedra with fixed vertex angles) and $\overline{\mathcal{M}}_{0,n}$. Is there some interpretation of $M_{1,n}$ as the space of triangulation of the torus with fixed vertex angles? Because that would be very cool. (Here are some notes of Rich Schwartz that I just found which might give a more comprehensible version of Thurston's paper: [4])

Higher dimensions? Rather than generalizing $\overline{\mathcal{M}}_{0,n}$ to $\overline{\mathcal{M}}_{g,n}$, what about generalizing to the moduli space of points on $\mathbb{CP}^2$, or $\mathbb{CP}^n$? I take it for granted that there's some theory of how to compactify this space and some extensive algebraic geometry literature on it. My question is: is there some interpretation which is relevant to stochastic geometry or statistical physics? Of course, the same question applies to other moduli spaces that people care about.

What's the geometry of spaces of random graphs? The discussion in this class has been about random polygons, which we could interpret as the space of piecewise-linear maps of the circle into $\mathbb{R}^3$ with a fixed number of pieces. However, many biologically significant polymers form more complicated graphs or networks. So we can think about piecewise-linear maps of a fixed arbitrary graph into $\mathbb{R}^3$ where we specify, say, how many pieces lie on each edge of the graph. The simplest example I can think of is a θ graph, which is just a two vertices with three edges connecting each of the two vertices. So: is there any special geometric structure on the space of all possible polygonal graphs with a fixed topological type and number of pieces along each edge? It's easy to cook up examples which are odd-dimensional, so these aren't necessarily symplectic or complex algebraic.

How to deal with thickness? While random polygons provide a reasonable statistical model for actual ring polymers, they're not particularly physically realistic: the polygons we've been considering allow two different edges to get arbitrarily close, or even to intersect, whereas of course real polymers have thickness. How can we incorporate a reasonable thickness constraint into either the symplectic or algebraic story? There is some sense in which this should be natural: remember that the action-angle coordinates on polygon space had the action coordinates given by lengths of chords in a given triangulation, meaning distances between certain vertices. So one can prevent certain vertices from closer to each other than ϵ simply by adding the constraint that these coordinates are not allowed to be smaller than ϵ. However, for n-gons there are only $n-3$ of these action coordinates, whereas there are $\binom{n}{2} - n = \frac{n(n-3)}{2}$ vertex-vertex distances that one would like to constrain (since distances between adjacent vertices are already fixed).

Hence, one would like some way of enforcing these constraints on all different triangulations (and hence, all choices of action-angle coordinates) at once. The triangulations of an n-gon are parametrized by a cluster algebra [3], so maybe this is the right formalism for incorporating these constraints?

Dynamics? Everything I've said so far is basically static: sampling random polygons, integrating over random polygons, etc. But in the physics, of course what one really cares about is the dynamics of the system, and how, for example, polymers change their conformations in response to energetic and entropic effects, etc. In principle one should be able to model the random motion of a polymer in solution by Brownian motion (or some other process) on $\mathrm{Pol}(n; \vec{1})$. How can we do this, preferably in our nice action angle coordinates? For example, is there a reasonable expression for the Laplacian in these coordinates? Relatedly, one would like to be able to compute distances and find geodesics between points in this space. Since $\mathrm{Pol}(n; \vec{1})$ is Kähler and hence has a Riemannian metric, this is in principle possible, but is it analytically or computationally tractable?

References

[1] Satyan L Devadoss. A space of cyclohedra. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 29(1):61–75, 2003.

[2] Yi Hu. Moduli Spaces of Stable Polygons and Symplectic Structures on $\overline{\mathcal{M}}_{0,n}$. *Compositio Mathematica*, 118(2):159–187, 1999.

[3] Bernard Leclerc and Lauren K Williams. Cluster algebras. *Proceedings of the National Academy of Sciences of the United States of America*, 111(27):9676–9679, July 2014.

[4] Richard Evan Schwartz. Notes on Shapes of Polyhedra. Preprint, `arXiv:1506.07252 [math.GT]`, 2015.

[5] William P Thurston. Shapes of polyhedra and triangulations of the sphere. *Geometry & Topology Monographs*, 1:511–549, 1998.