Math 676: Day 37

Recall from last time that $\overline{Pol}(n;\vec{r}) \cong \overline{M}_{0,n}$

This suggests the following conjecture:

Conjecture (Hu): The Kähler cone $K(\overline{M}_{0,n})$ of $\overline{M}_{0,n}$ (for $n \geq 5$) is isomorphic to a subcone of

$$\mathcal{C} := \left\{ (\vec{r}, \vec{\varepsilon}) \in C^* \times \mathbb{R}_+^{2^{n-1}-1-n-\frac{n(n-1)}{2}} \;\middle|\; 0 < \varepsilon_J < 2\min\{r_j\}_{j \in J},\ J \text{ a relevant subset of } [n] \right\}$$ ← positive convex cone

where C^* is the cone over the/a chamber containing $(\frac{2}{n}, \dots, \frac{2}{n}) \in \Delta(2,n)$.

A remark which I only partially understand:

Dolgachev-Hu, Pub. Math. IHES 87 (1998), 5-56

Remark 9.4. By 3.3.21 of [2], $C(\mathbb{D}_2^n)$ is the G-ample cone for both the PGL(2)-action on $(\mathbb{P}^1)^n$ and the maximal torus action on the Grassmannian $G(2, \mathbb{C}^n)$. It is known that the Chow quotients of these two actions can be identified with $\overline{\mathcal{M}}_{0,n}$. Thus the above conjecture would establish an interesting connection between the G-ample cone of projective G-variety and the ample cone of its Chow quotient. This and moreover the case for a general algebraic group action call for further investigation.

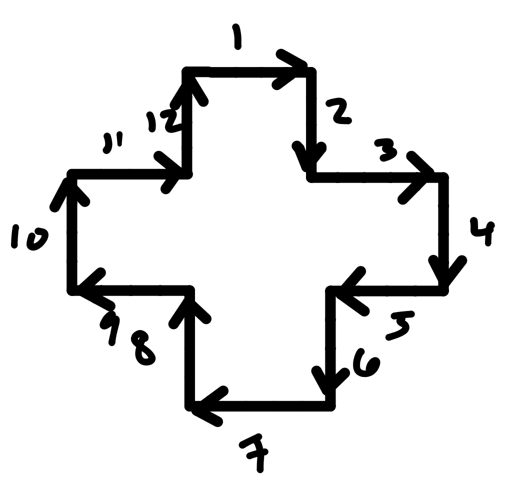
Last thing: think about the map $\overline{Pol}(n;\vec{r}) \to Pol(n;\vec{r})$ which forgets all bubbles. The claim is that for generic $\vec{r}$ this is an explicit iterated blow-up, & for $\vec{r}$ on a wall this is a composition of a canonical resolution of singularities & an explicit iterated blow-up.

Basically, this will be immediate once we identify the subvarieties that are getting blown up.

What should be getting blown up in $Pol(n;\vec{r})$?

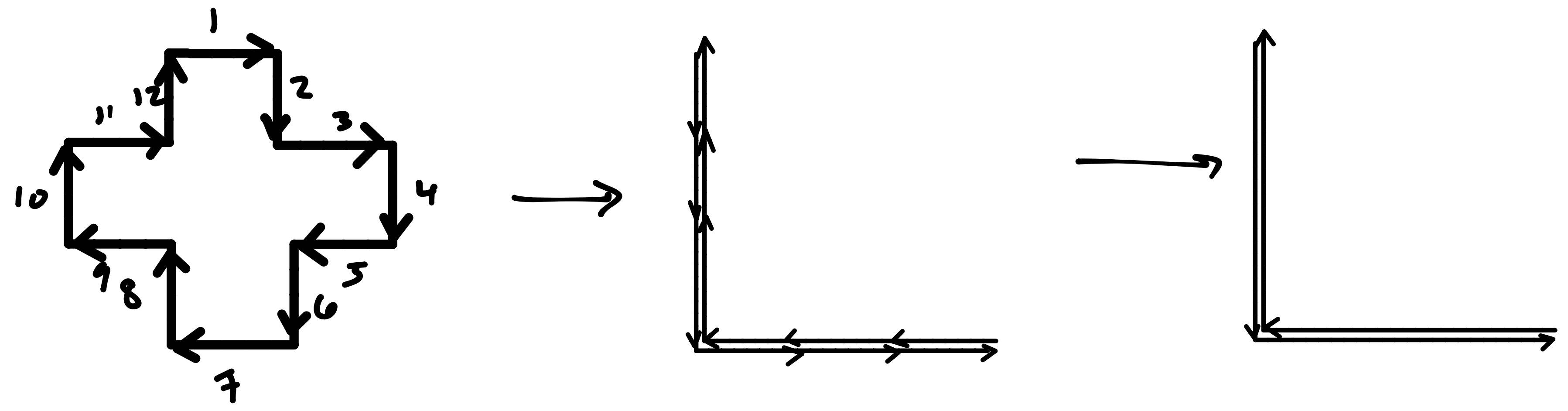
Surely the subvarieties of polygons which degenerate at specified collections of edges. So, to that end, if $F[n]$ is the set of all partitions of $[n]$ & $\alpha \in F[n]$ is of the form $\alpha = I_1 \sqcup I_2 \sqcup \dots \sqcup I_k$, then define Y_α to be the set of all polygons $P \in Pol(n;\vec{r})$ s.t. the edges e_{I_s} $(1 \leq s \leq k)$ are parallel.

Ex: with $n = 12$, $\vec{r} = \vec{1}$ & $\alpha = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 = \{1,3,11\} \sqcup \{2,4,6\} \sqcup \{5,7,9\} \sqcup \{8,10,12\}$, a typical element of Y_α is



In turn, a generic elt of $Pol(12;\vec{1})$ belongs to Y_β w/ $\beta=\{1\}\sqcup\{2\}\sqcup\dots\sqcup\{12\}$

Clearly, this $Y_\beta \cong Pol(12;\vec{1})$; in general, $Y_\alpha \cong Pol(k;\vec{r}_\alpha)$ where $\vec{r}_\alpha=\left(\sum_{i\in I_1} r_i,\dots,\sum_{i\in I_k} r_i\right)$. For example, the first ex. can be pointed to get



$\in Pol(4;\vec{3})$

Since $\{1\}\sqcup\dots\sqcup\{n\}$ corresponds to the generic n-gons (& hence is in some sense the "biggest" partition), this suggests we should make $F[n]$ a poset ordered by reverse refinement; i.e., for $\alpha,\beta\in F[n]$ w/ $\alpha=I_1\sqcup\dots\sqcup I_k$, $\beta=J_1\sqcup\dots\sqcup J_m$, we say $\alpha\preceq\beta \iff \forall\, 1\le s\le m$ we have $J_s\subseteq I_t$ for some $1\le t\le k$.

Now we see that if $\alpha\preceq\beta$ then we have $Y_\alpha\subseteq Y_\beta$. Moreover, the Y_α are generally smooth unless $\vec{r}_\alpha$ lies on a wall of $C(\Delta(2,n))$; but in that case Y_α contains a smooth dense open subset Y_α° consisting of those elts. for which the e_{I_s} are parallel to each other but to no other edges (notice that the lined n-gons only appear as elts. of Y_α° when $\alpha=I\sqcup I^c$, but there $Y_\alpha^\circ=Y_\alpha=$ single point, which is trivially a smooth 0-mfd)

So then $Pol(n;\vec{r})=\bigcup_{\alpha\in F[n]} Y_\alpha^\circ$ is a smooth stratification of $Pol(n;\vec{r})$, & the theorem is:

Thm (Hu): Assume $Pol(n;\vec{r})$ is smooth. Then the projection $\overline{Pol}(n;\vec{r})\to Pol(n;\vec{r})$ is the iterated blowup of $Pol(n;\vec{r})$ along (the proper transforms of) all the smooth closed strata Y_α in the order dictated by the partial order on $F[n]$, starting from the smallest ones.

In general, if $Pol(n;\vec{r})$ is singular (e.g. $\vec{r}=\vec{1}$ & n even), then it has isolated singularities corresponding to lined gons which, as discussed above, form one of the smallest strata $Y^\circ_{I\sqcup I^c}$. The claim is that these singularities admit canonical resolutions, & hence:

Thm (Hu): Assume $Pol(n;\vec{r})$ is singular. Then $\overline{Pol}(n;\vec{r})\to Pol(n;\vec{r})$ is the composite of the canonical resolution of singularities followed by the iterated blowups of the resulting resolution along (the proper transforms of) all other smooth closed strata in the order dictated by the partial order, starting from the smallest.