Math 676: Day 36

Now, $(P,P')\in Pol(n;\vec r)\times Pol(|J|+1;\vec r_J)$ is called a bubble pair if P degenerates at e_J & then P' is called a bubble of P.

Note that P' can't degenerate at the last edge & (P,P') being a bubble pair implies $\sum_{j\in J} r_j \le \sum_{i\notin J} r_i$

Given $\vec r$ in the interior of $\Delta(2,n)$, we fix a uniform ε once & for all which will play the role of ε_J for each possible J that arises in a bubble pair (e.g., $\varepsilon=\min\{r_1,\dots,r_n\}$).

Def: A stable n-gon w.r.t. $\vec r$ & ε is a collection of labeled (but not ordered) polygons

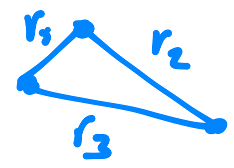
$$\mathbb{P}:=(P_0,P_1,\dots,P_m)\in Pol(n;\vec r)\times Pol(|J_1|+1;\vec r_{J_1})\times\dots\times Pol(|J_m|+1;\vec r_{J_m})$$

s.t.

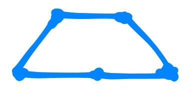
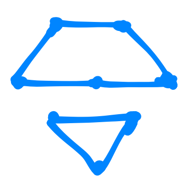
① If $J_t \subsetneq J_s$, then P_t is a bubble of P_s.

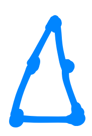
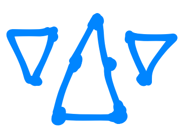
② If P_h does not have a bubble, then it is generic (i.e., $P_h\in Pol(|J_h|+1;\vec r_{J_h})^{\circ}$).

Denote the set of stable n-gons w/ edgelength vector $\vec r$ & choice of ε by $\overline{Pol}(n;\vec r,\varepsilon)$ or just $\overline{Pol}(n;\vec r)$.

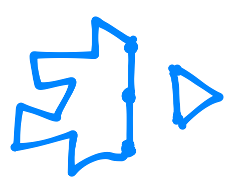
Ex: For any point $\vec r$ in the interior of $\Delta(2,3)$, the space $Pol(3;\vec r)$ consists of a single stable triangle 

Ex: Consider $Pol(5;\vec 1)$, the space of equilateral pentagons. $\vec 1\in C(\Delta(2,5))$ (it sits on the line containing $(\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5})$)

Then an element can degenerate only by having 2 parallel edges: , which we stabilize by forming the unique triangle w/ sidelengths 1, 1, & $2-\varepsilon=2-1=1$: 

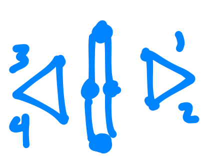
Even worse, we could have , which stabilizes to 

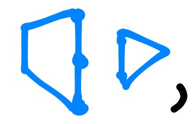
Notice that if a polygon degenerates at e_J w/ $|J|=2$, then the corresponding bubble is rigid: just a single triangle,

so, e.g., the subset of $\overline{Pol}(n;\vec r)$ corresponding to  is just a copy of $Pol(n-1;(2,r_3,r_4,\dots,r_n))$

↑ up to permutation

In particular, notice that $\overline{Pol}(4;\vec 1)$ has 3 special points, corresponding to the possible permutations of edges in



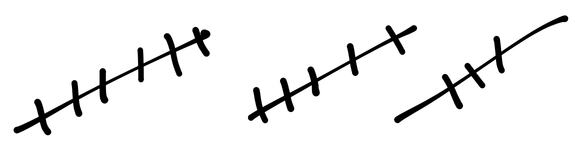
& that the boundary divisors of $\overline{Pol}(5;\vec{1})$ look like , which is a copy of $\underbrace{\overline{Pol}(4;(2,1,1,1))}_{\cong \mathbb{CP}^1} \times \underbrace{\overline{Pol}(3;(1,1,1))}_{=\{*\}}$

since $(2,1,1,1) \in C(\Delta_1) \subseteq C(\Delta(2,4))$

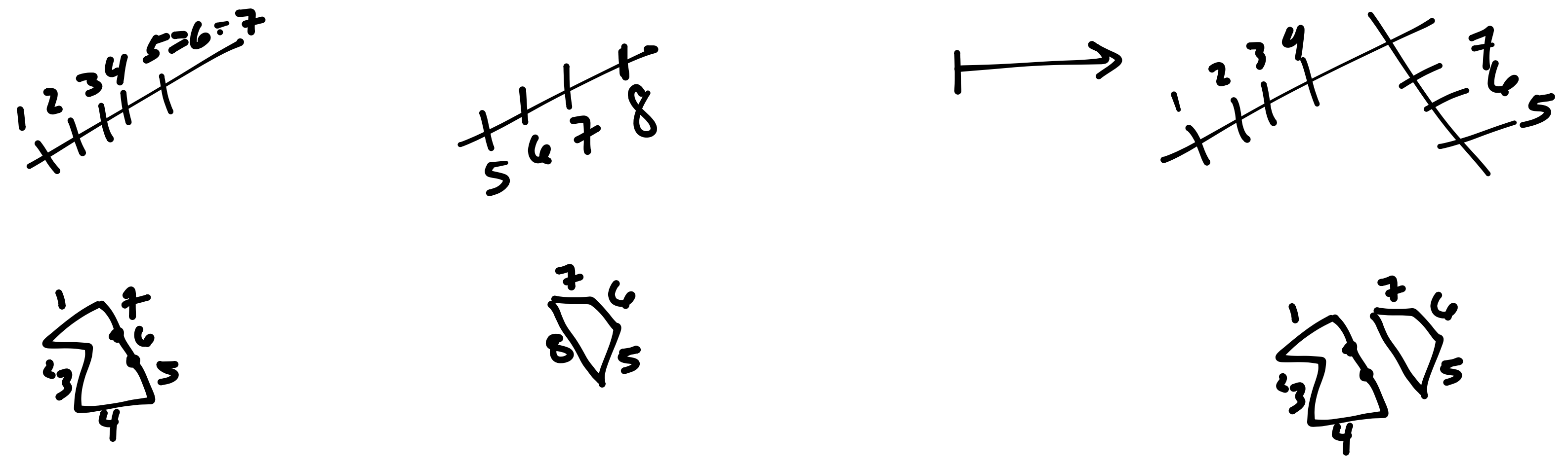
which are certainly our 10 special lines in $\overline{Pol}(5;\vec{1}) \cong \overline{M}_{0,5}$.

Thm (Hu): Let $\vec{r}$ be a point in the interior of $C(\Delta(2,n))$ & choose ε as before. Then $\overline{Pol}(n;\vec{r})$ & $\overline{M}_{0,n}$ are biholomorphic. In particular, the complex structure on $\overline{Pol}(n;\vec{r})$ is independent of $\vec{r}$ & ε (though the Kähler structure is not). Moreover, $\overline{Pol}(n;\vec{r}) \setminus Pol(n;\vec{r})^\circ$ is a divisor w/ normal crossings.

Idea: Obviously, $Pol(n;\vec{r})^\circ$ is biholomorphic to $M_{0,n}$, so the only problem is to extend to the boundary.

Now, given $P=(P_0,\dots,P_m) \in Pol(n;\vec{r}) \times Pol(|J_1|+1;\vec{r}_{J_1}) \times \dots \times Pol(|J_m|+1;\vec{r}_{J_m})$, we get $\mathbb{X}=(X_0,\dots,X_m)$, a collection of pointed curves. 

If (P_s,P_t) are a bubble pair, then we join X_s & X_t at the coinciding points of X_s & at the long edge in X_t



One then checks that this map is holomorphic & bimeromorphic, & hence biholomorphic. ∎

At least for $\vec{r}$ away from the walls, GAGA theorem $\Rightarrow$ $\overline{Pol}(n;\vec{r})$ & $\overline{M}_{0,n}$ are isomorphic as projective varieties as well.