Math 476: Day 35

Recall that we're thinking about $\mathrm{Pol}(n;\vec{r})$, where $\vec{r}=(r_1,\dots,r_n)$ w/ $\sum r_i = 2$ & $0 \le r_i \le 1$ $\forall i$, so $\vec{r} \in \Delta(2,n)$.

Also, $\Delta(2,n)$ is broken up into chambers by walls

$$W_J := \left\{ (r_1,\dots,r_n) \in \Delta(2,n) : \sum_{i\in J} r_i = 1 \right\} \quad \text{for } J \subseteq [n]$$

where the interior walls are those corresponding to $2 \le |J| \le n-2$.

For $\vec{r}, \vec{r}\,'$ in the same chamber, $\mathrm{Pol}(n;\vec{r})$ & $\mathrm{Pol}(n;\vec{r}\,')$ are biholomorphic but not necessarily isomorphic as Kähler manifolds.

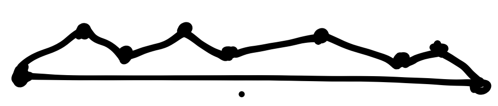
Now, the key thing we'll need is the following:

Def: For $i \in [n]$, the *favorable chamber* Δ_i is the unique maximal chamber in $\Delta(2,n)$ containing the simplex facet $W_{\{i\}}$.

The chamber Δ_i is characterized in the following way: $\vec{r} \in \Delta_i \iff \sum_{j\in J} r_j < 1$ $\forall\, J \subsetneq [n]$ w/ $i \notin J$ & $|J| < n-1$.

Equivalently, $\sum_{j\in J} x_j + x_i > 1$ $\forall\, J \subsetneq [n]$ w/ $i \notin J \iff x_j + x_i > 1$ $\forall\, j \neq i$

Geometrically, if $\vec{r} \in \Delta_i$, then a polygon in $\mathrm{Pol}(n,\vec{r})$ has a very long i^{th} edge, so long that the dot products w/ all other edges must be negative



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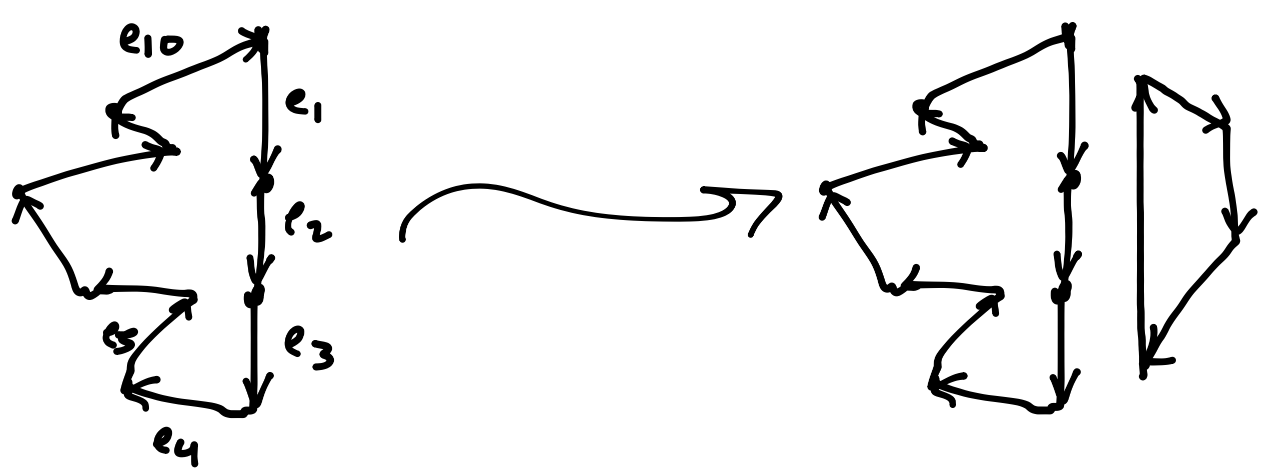
Prop (Hu, Hausmann): For $\vec{r}$ in the interior of Δ_i, $\mathrm{Pol}(n,\vec{r}) \cong \mathbb{CP}^{n-3}$.

Def: A polygon is said to *degenerate* at a set of edges $e_I := \{e_i\}_{i\in I}$ w/ $|I|>1$ if the $\{e_i\}_{i\in I}$ are all parallel & no other edges are parallel to them. An edge is *degenerate* if it belongs to a degenerate set of edges.

Now, remember that the idea is to bubble off a new polygon whenever we have degenerate edges



If e_J is the set of degenerate edges, then we let the sidelengths of the bubble to be $\vec{r}_{J,\varepsilon} := (r_{j_1},\dots,r_{j_m}, \sum_{j\in J} r_j - \varepsilon)$ for ε chosen small enough that $\vec{r}_{J,\varepsilon}$ is in the chamber $C(\Delta_{m+1})$ of $C(\Delta(2,m+1))$.

Prop: If $\vec{r}$ is in the interior of $C(\Delta(2,m))$, then $(r_1,\dots,r_m, \sum r_i - \varepsilon)$ is in $C(\Delta_{m+1}) \iff \varepsilon < 2\min(r_1,\dots,r_m)$.

Proof: To be in $C(\Delta_{m+1})$ we need $\sum_{j\in J} r_j < \frac{1}{2}\left(\sum r_i + (\sum r_i - \varepsilon)\right)$ $\forall\, J \subsetneq [m]$ or, equivalently,

$$\frac{\varepsilon}{2} < \sum_{i\notin J} r_i \quad \text{for all } J \subsetneq [m],$$

which happens $\iff \varepsilon < 2\min\{r_1,\dots,r_m\}$, as desired. ∎

In particular, combining w/ the prev. prop., we see that for any $J \subseteq [n]$ w/ $2 \le |J| \le n-2$ & any $0 < \varepsilon_J < 2\min_J\{r_j\}$

the vector $\vec{r}_{J,\varepsilon_J} = (r_J, \sum_J r_j - \varepsilon_J)$ lies in a favorable chamber, & hence $Pol(|J|+1; \vec{r}_{J,\varepsilon_J}) \cong \mathbb{CP}^{|J|-2}$.

Moreover, no polygon in $Pol(|J|+1; \vec{r}_{J,\varepsilon_J})$ ever degenerates at the last edge.

To be specific, we can certainly choose $\varepsilon_J = \min_J\{r_j\}$ & so, for example, w/ equilateral polygons we can get a uniform ε which is just the length of a side (either 1 or $\frac{2}{n}$, depending on the situation) & we just write $\vec{r}_J$ rather than $\vec{r}_{J,\varepsilon_J}$.