Math 676: Day 34

A polygonal interpretation of $\overline{M}_{0,n}$

For this discussion, we follow: Yi Hu, "Moduli spaces of stable polygons & symplectic structures on $\overline{M}_{0,n}$", Compositio Math. 118 (1999), 159-187.

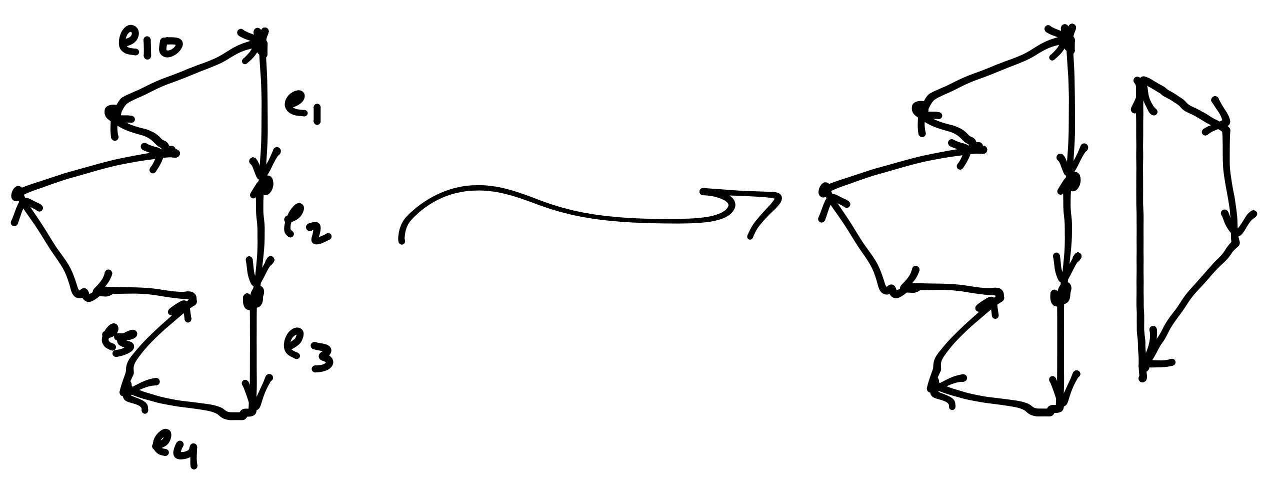
Basic idea: We already know that the open, dense subset $Pol(n)^\circ \subseteq Pol(n)$ w/ no two parallel edges is isomorphic to $M_{0,n}$.

Now, the symplectic reduction/GIT quotient gives one possible compactification, $Pol(n)$.

The Deligne-Mumford compactification gives $\overline{M}_{0,n}$, which is richer (& of course we also get all the Hassett spaces $M_{0,A}$ for $|A|=n$)

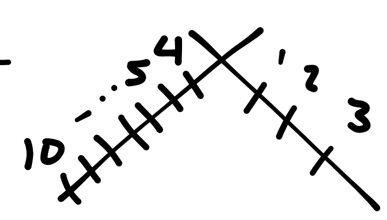
Now, the idea is to recapitulate the construction of $\overline{M}_{0,n}$ as stable nodal curves in polygonal terms.

When edges become parallel, we want to bubble off a new polygon:



The tricky bit is to choose the long edge in the bubbled-off polygon in such a way that we have minimal flexibility.

When we do that, we get a nodal interpretation of the above w/ a nodal curve



The space of stable polygons depends on a vector $\vec{\varepsilon}$, so we call it $Pol(n)_{\vec{\varepsilon}}$.

There's a natural projection $\pi_{\vec{\varepsilon}} : Pol(n)_{\vec{\varepsilon}} \to Pol(n)$ by forgetting the bubbles.

Thm (Hu): $\pi_{\vec{\varepsilon}}$ is holomorphic & bimeromorphic. When $Pol(n)$ is smooth, it is the iterated blowup of $Pol(n)$ along some explicitly described smooth subvarieties. When $Pol(n)$ is singular, it is the composite of a canonical resolution of singularities followed by explicit iterated blowups.

This generalizes both the Keel blowup representation $\overline{M}_{0,n} \to (\mathbb{P}^1)^{n-3}$ & the Kapranov blowup representation $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$.

In general, we will need to deal w/ spaces of non-equilateral n-gons, so . . .

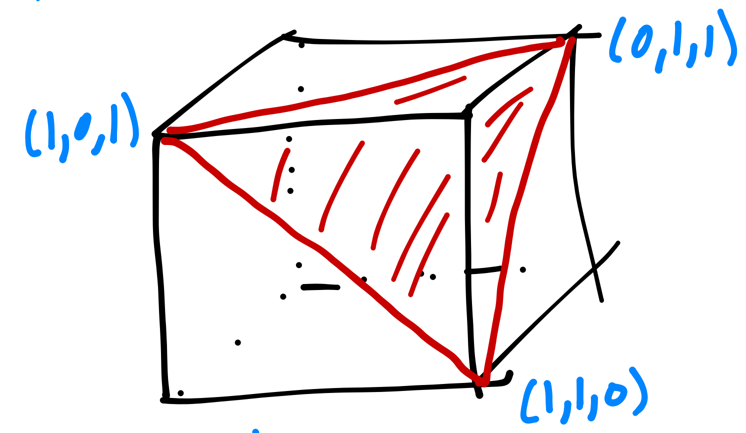
Let $\mathrm{Pol}(n,\vec r)$ be the space of $\overbrace{\text{closed polygons}}^{\text{up to translation \& rotation}}$ w/ edgelengths specified by the vector $\vec r=(r_1,\dots,r_n)$ s.t. $0\le r_i\le 1$ $\forall i$, $\sum r_i=2$.

So the space of equilateral n-gons is $\mathrm{Pol}(n)=\mathrm{Pol}(n,\vec 1)$.

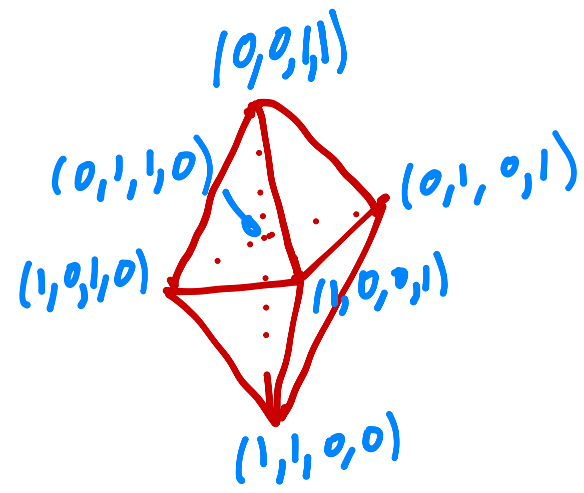
The possible edgelength vectors $\vec r$ live in the second hypersimplex

$$\Delta(2,n):=\left\{(r_1,\dots,r_n)\in\mathbb{R}^n : 0\le r_i\le 1,\ \sum r_i=2\right\}$$

Ex: $\Delta(2,3)$ is an upside-down simplex:



$\Delta(2,4)$ is a regular octahedron



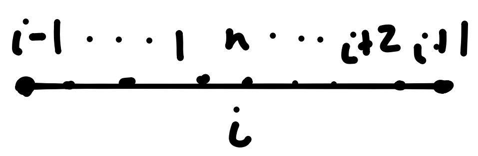
Now, $\Delta(2,n)$ is broken up into chambers, w/ walls

$$W_J:=\left\{(r_1,\dots,r_n)\in\Delta(2,n) : \sum_{i\in J} r_i=1\right\} \quad \text{for } J\subseteq[n]$$

corresponding to choices of edgelengths for which it is possible to have lined polygons (& hence $\mathrm{Pol}(n,\vec r)$ is singular).

If $|J|=1$ or $|J^c|=1$, then W_J is a facet of $\Delta(2,n)$ which is a simplex (same thing $\sum_{i\in J^c} r_i=1$)

corresponding to spaces of polygons like



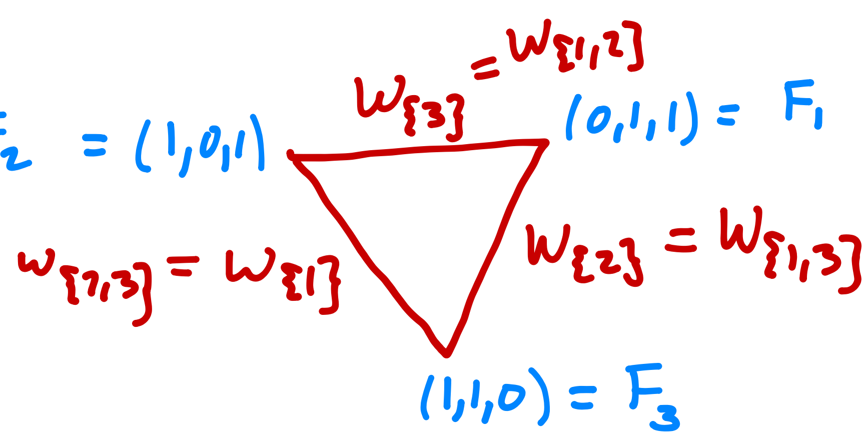
Of course, for, say, $\vec r\in W_{\{i\}}$, $\mathrm{Pol}(n,\vec r)=\{*\}$ is a single point.

Other facets are of the form $F_i=\{(r_1,\dots,r_n)\in\Delta(2,n): r_i=0\}$, which is a copy of $\Delta(2,n-1)$.

The other walls, W_J for $2\le|J|\le n-2$, are interior walls.

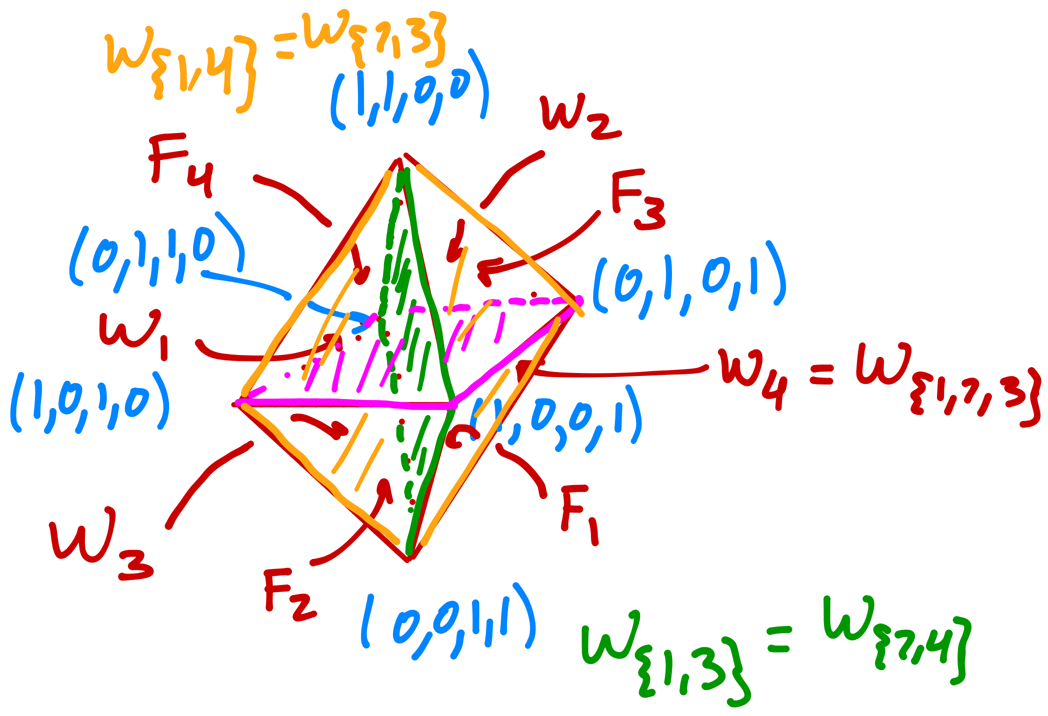
When $\vec r$, $\vec r'$ are in the same chamber, $\mathrm{Pol}(n,\vec r)$ & $\mathrm{Pol}(n,\vec r')$ are biholomorphic.

Ex: On $\Delta(2,3)$ there are no interior walls & the F_i are points.



Ex: On $\Delta(2,4)$

$W_{\{3,4\}} = W_{\{1,2\}}$



Notice that the edgelength vector $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ of the equilateral 4-gons lies at the intersection of all 3 internal walls.