Math 476: Day 31

We saw last time that the image of the moment map of a Hamiltonian torus action on a cpct symplectic mfd is a convex polytope, called the

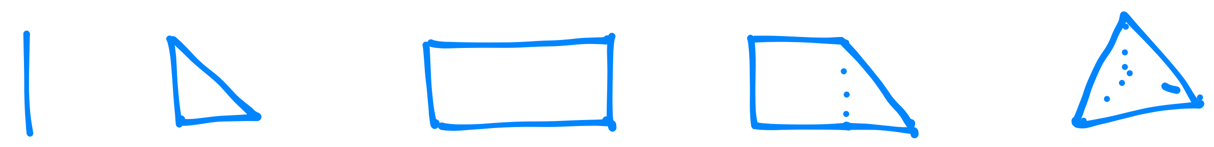
moment polytope.

In fact, when M is toric, we get even more: the moment polytope is Delzant, meaning it is a convex polytope in $\mathbb{R}^n$ s.t.

① simple, meaning that there are n edges meeting at each vertex

② rational, meaning all edges meeting at a vertex p are of the form $p+tu_i$, $t\geq 0$, $u_i\in\mathbb{Z}^n$.

③ smooth, meaning that at each vertex the edge vectors $u_1,\dots,u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Examples: 

Non-examples: 

not smooth
$(0,-1)$ & $(2,-1)$
not a $\mathbb{Z}$-basis for $\mathbb{Z}^2$

not simple: 4 edges at each vertex

But then the amazing fact is that understanding toric symplectic mfds is purely combinatorial:

Thm (Delzant): There is a one-to-one correspondence

$$\{\text{toric symplectic mfds}\}\xleftrightarrow{1\text{-}1}\{\text{Delzant polytopes}\}$$

$$(M^{2n},\omega,T^n,\mu)\longmapsto \mu(M)$$

Basic idea: Let v_i be the primitive outward normals of a Delzant polytope $\Delta=\{x\in(\mathbb{R}^n)^*\mid \langle x,v_i\rangle\leq\lambda_i,\ i=1,\dots,d\}$

Then $e_i\mapsto v_i$ defines a surjective map $\mathbb{Z}^d\to\mathbb{Z}^n$, & hence a map $\pi: T^d=\mathbb{R}^d/\mathbb{Z}^d\to T^n=\mathbb{R}^n/\mathbb{Z}^n$.

Let $N=\ker\pi$ & now consider $(\mathbb{C}^d, \frac{i}{2}\sum dz_k\wedge d\bar{z}_k)$ w/ the std. T^d action: $(e^{i\theta_1},\dots,e^{i\theta_d})\cdot(z_1,\dots,z_d)=(e^{i\theta_1}z_1,\dots,e^{i\theta_d}z_d)$,

which has moment map $\mu:\mathbb{C}^d\to(\mathbb{R}^d)^*$ given by $\varphi(z_1,\dots,z_d)=-\frac{1}{2}(|z_1|^2,\dots,|z_d|^2)+\text{const}$,

where we can pick the const to be $(\lambda_1,\dots,\lambda_d)$.

Now, the restriction of the action to the subgroup $N<T^d$ is also Hamiltonian, & the moment map is the composition w/ the projection $(\mathbb{R}^d)^*=(\mathfrak{t}^d)^*\xrightarrow{i^*}\mathfrak{n}^*$; i.e. $\mu:\mathbb{C}^d\to\mathfrak{n}^*$ is given by $\mu=i^*\circ\varphi$.

But then one can show that N acts freely on $\mu^{-1}(\vec{0})$ & that $M_{\Delta} := \mu^{-1}(\vec{0})/N = \mathbb{C}^d /\!\!/_{\vec{0}} N$ is toric.

The final piece is the connection to measure theory:

Thm (Duistermaat-Heckman): If $(M^{2n}, \omega, T^m, \mu)$ is a Hamiltonian T^m-space w/ Liouville measure m given by

$m(A) := \int_U \frac{\omega^n}{n!}$, then the pushforward measure $\mu_* m$ on the moment-polytope is a piecewise polynomial multiple of Lebesgue (a.k.a. uniform) measure λ. More precisely, for any Borel subset U of the moment polytope,

$$\mu_* m(U) = \int_U f \, d\lambda$$

where f is a polynomial of degree $\leq n-m$ on each region of the polytope consisting of regular values

In particular, when $m=n$ & the mfd is toric, f is the constant function $(2\pi)^n$.

Ex: For the std. Hamiltonian S^1 action on (S^2, ω_{std}), $\mu_* m = 2\pi\lambda$. Indeed, for $[a,b] \subseteq [-1,1]$,

$$\lambda([a,b]) = b-a$$

&

$$\mu_* m([a,b]) = m(\mu^{-1}([a,b])) = m(\{(x,y,z) \in S^2 : a \leq z \leq b\}) = \int_{\{a \leq z \leq b\}} d\theta \wedge dz = \int_{z\in[a,b]} \int_{\theta\in[0,2\pi]} d\theta \, dz$$

$$= \int_{z\in[a,b]} 2\pi \, dz = 2\pi(b-a)$$

Of course, this extends to all other Borel sets.

Therefore, provided we can find coordinates on the torus fibers, M & $T^n \times P$ are measure-theoretically isomorphic.

torus ↗ ↖ moment polytope