Math 676: Day 30

Last time, we saw that the equilateral polygon space $\mathrm{Pol}(n)$ sits at the bottom of a chain of iterated symplectic reductions or GIT quotients:

$$\begin{tikzcd}
\mathbb{C}^{n\times 2} \arrow[d, "/\!/_0 (U(1))^n \text{ or } /\!/(\mathbb{C}^*)^n"] \\
(S^2)^n \cong (\mathbb{CP}^1)^n \arrow[d, "/\!/_0 SO(3) \text{ or } /\!/ PSL(2,\mathbb{C})"] \\
\mathrm{Pol}(n)
\end{tikzcd}$$

So now it's reasonable to ask: what happens if you take the quotients in the opposite order?

Of course, $PSL(2,\mathbb{C})$ doesn't really act on $\mathbb{C}^{n\times 2}$, but $PSL(2,\mathbb{C}) = PGL(2,\mathbb{C})$, & $GL(2,\mathbb{C})$ certainly acts on the right.

Now, $\dim_{\mathbb{C}} PSL(2,\mathbb{C}) = 3$ & $\dim_{\mathbb{C}} GL(2,\mathbb{C}) = 4$, so it turns out you lose a factor from the torus action

& the natural iterated quotient to take is

$$\begin{tikzcd}
\mathbb{C}^{n\times 2} \arrow[d, "/\!/_0 U(2) \text{ or } /\!/ GL(2,\mathbb{C})"] \\
? \arrow[d, "/\!/_0 (U(1))^{n-1} \text{ or } /\!/(\mathbb{C}^*)^{n-1}"] \\
\mathrm{Pol}(n)
\end{tikzcd}$$

The fact that you still get $\mathrm{Pol}(n)$ at the bottom is a deep fact, known as the Gelfand-McPherson correspondence in GIT, where it's a general fact: changing the order of iterated GIT quotients doesn't affect the end result.

The fact that symplectic reductions still give the right answer is known in this particular instance b/c of work of Hausmann-Knutson, but doesn't seem to be known in general (except, obviously, when it can be deduced from Kirwan-Kempf-Ness, but not all symplectic reductions can be realized as GIT quotients).

In the diagram, it's fairly easy to guess that the missing space is the Grassmannian $G_2(\mathbb{C}^n)$: we think of elts of $\mathbb{C}^{n\times 2}$ as pairs of vectors in $\mathbb{C}^n$, which generically span a 2-plane & the right $GL(2,\mathbb{C})$ action just gives all possible changes of basis for the 2-plane.

So, ultimately, we get the diagram

$$\begin{tikzcd} & \mathbb{C}^{n\times 2} \arrow[dl, dash, "/\!\!/_\theta U(2)"'] \arrow[dr, dash, "/\!\!/_\theta (U(1))^n"] & \\ G_2(\mathbb{C}^n) \arrow[dr, dash, "/\!\!/_\theta (U(1))^{n-1}"'] & & (S^2)^n \arrow[dl, dash, "/\!\!/_\theta SO(3)"] \\ & Pol(n) & \end{tikzcd}$$

Hausmann-Knutson & Howard-Manon-Millson gave a polygonal interpretation of $G_2(\mathbb{C}^n)$, which consists of all spin-framed polygons in $\mathbb{R}^3$ of total perimeter 2 (but variable edgelengths) up to translation & rotation.

One additional useful structure on $Pol(n)$: it's toric, meaning it has a Hamiltonian action of a half-dimensional torus.

Ex: S^2 is toric: we know the S^1-action which rotates around a fixed axis is Hamiltonian, & $\dim S^1 = 1 = \frac{1}{2}\dim S^2$

Ex: $S^2 \cong \mathbb{C}P^1$; more generally, $\mathbb{C}P^n$ is toric: define a $T^n = (U(1))^n$ action by

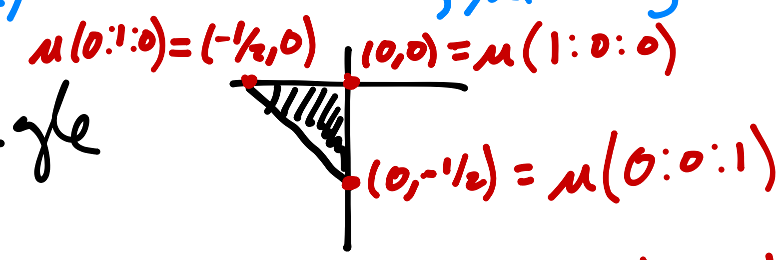
$$(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_0 : z_1 : \dots : z_n) := (z_0 : e^{i\theta_1} z_1 : \dots : e^{i\theta_n} z_n),$$

which turns out to be Hamiltonian w/ moment map $\mu(z_0 : \dots : z_n) = -\frac{1}{2}\left(\frac{|z_1|^2}{\Sigma |z_i|^2}, \dots, \frac{|z_n|^2}{\Sigma |z_i|^2}\right)$

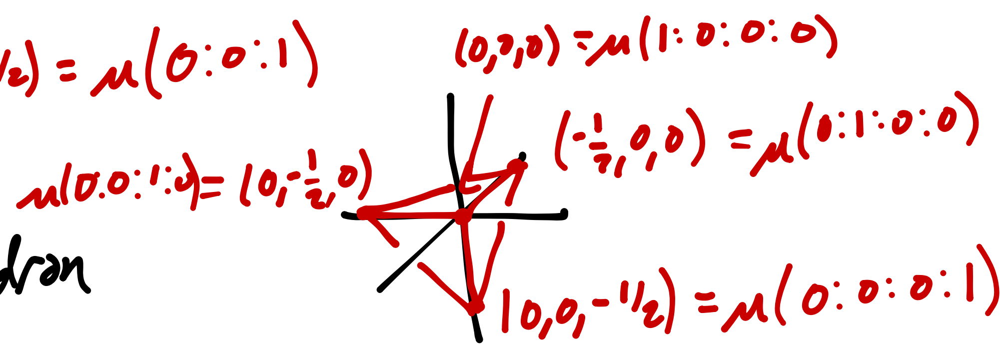
In the second example, the fixed pts & image of the moment map are:

$n=1$: $(1:0)$ & $(0:1)$ & the interval $[-\frac{1}{2}, 0] = [\mu(1:0), \mu(0:1)]$

$n=2$: $(1:0:0)$, $(0:1:0)$, & $(0:0:1)$ & the triangle



$n=3$: $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$, $(0:0:0:1)$ & the tetrahedron



etc.

The same holds in general:

Thm (Atiyah, Guillemin-Sternberg): Let (M,ω) be a compact, connected symplectic mfld & suppose the torus T^m acts in a Hamiltonian way w/ moment map $\mu : M \to \mathbb{R}^m$. Then

1. the level sets of μ are connected
2. the image of μ is convex
3. the image of μ is the convex hull of the images of the fixed points of the action.