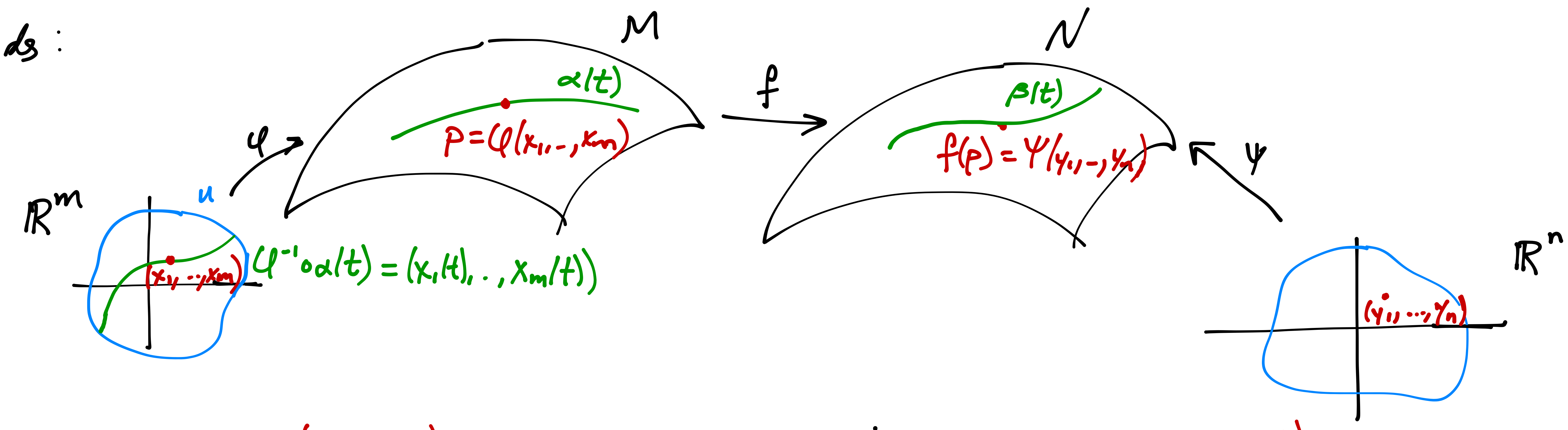
Math 676: Day 3

Def: Suppose M^m & N^n are manifolds & $f: M \to N$ smooth. For $p \in M$, the **differential** of f at p is the linear map $df_p : T_pM \to T_{f(p)}N$ given by:

For $v \in T_pM$, choose $\alpha: (-\varepsilon, \varepsilon) \to M$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$. Let $\beta = f \circ \alpha$ & define

$$df_p(v) := \beta'(0).$$

In coords:



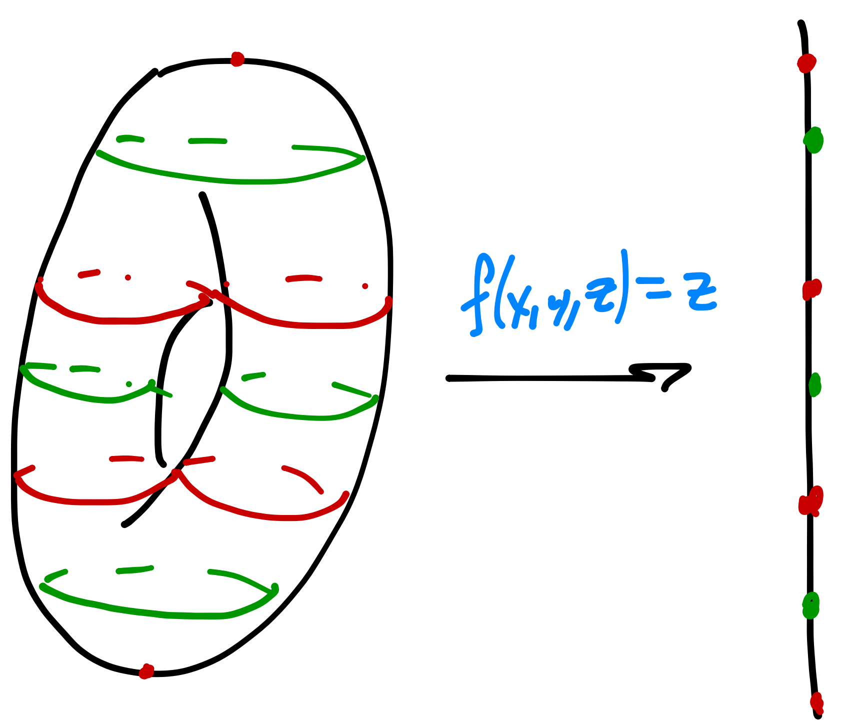
$$(y_1, \dots, y_n) = \psi^{-1} \circ f \circ \varphi(x_1, \dots, x_n) = (y_1(x_1, \dots, x_m), \dots, y_n(x_1, \dots, x_m))$$

Then

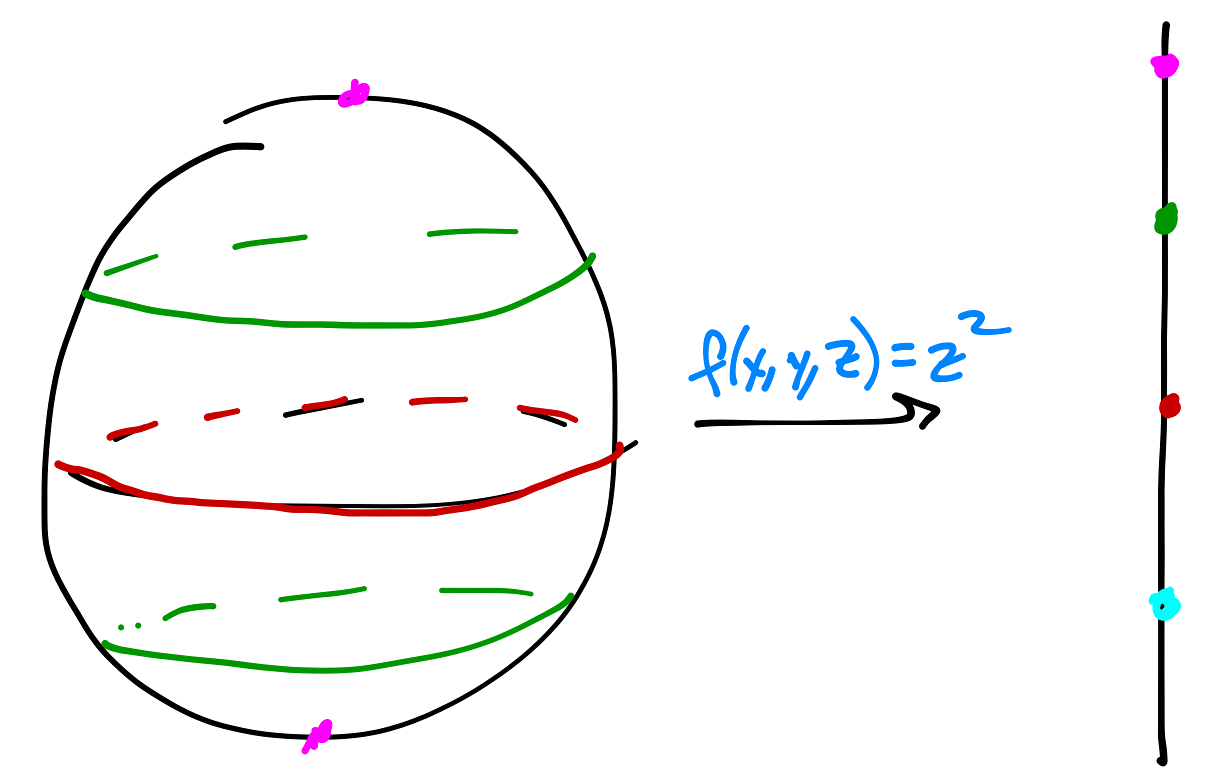
$$d(\psi^{-1})_{f(p)} \beta'(0) = \left(\sum \frac{\partial y_1}{\partial x_i} x_i'(0), \dots, \sum \frac{\partial y_n}{\partial x_i} x_i'(0) \right) = \underbrace{\left(\frac{\partial y_i}{\partial x_j} \right)}_{\substack{n \times m \text{ matrix} \\ \text{"}df_p\text{"}}} \underbrace{\left(x_j'(0) \right)}_{\substack{m\text{-element column vector} \\ \text{"}v\text{"}}}$$

Def: Let $f: M^m \to N^n$ be smooth. A point $p \in M$ is a **critical point** of f if $df_p : T_pM \to T_{f(p)}N$ is not surjective; then $f(p)$ is a **critical value** of f. A point $q \in N$ which is not a critical value is a **regular value**.

Cliché Example:



Ex:



Thm: If $q \in N$ is regular, then $f^{-1}(q)$ is a smooth submanifold of M of dimension $m - n$.

Proof: Inverse Function Thm. ▨

Ex: Define $f: Mat_{n\times n}(\mathbb{R}) \to \{\text{symmetric } n\times n \text{ matrices}\}$ by $f(A) = AA^T$. Then df is always surjective & $f^{-1}(I) = O(n)$ is a smooth submanifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Ex: Remember our map $\mu: \underbrace{S^{d-1} \times \dots \times S^{d-1}}_{n} \to \mathbb{R}^d$ given by $\mu(\vec{e}_1, \dots, \vec{e}_n) = \sum_{i=1}^{n} \vec{e}_i$. Then $\vec{0} \in \mathbb{R}^d$ is a regular value when $n \geq 2$, so the polygon space $\mu^{-1}(\vec{0})$ is a smooth submanifold of codimension d.

Vector Fields:

Def: A **vector field** X on a smooth mfd M is a smooth section of the tangent bundle TM; i.e., a smooth map $X: M \to TM$. Given how we've defined tangent vectors, this means a vector field is just a linear derivation on $C^\infty(M)$.

Note: If $\varphi: U \subseteq \mathbb{R}^n \to M$ is a coord. chart at $p \in M$, then

$$X(p) = \sum a_i(p) \frac{\partial}{\partial x_i},$$ where each $a_i: U \to \mathbb{R}$ is smooth & $\{\frac{\partial}{\partial x_i}\}$ is the basis of T_pM associated w/ (U, φ).

- As a map $C^\infty(M) \to C^\infty(M)$, a vector field acts as

$$(Xf)(p) = \sum a_i(p) \frac{\partial f}{\partial x_i}\Big|_p$$

which notice really is a smooth function on M.

Def: If X & Y are smooth vector fields on M, the **Lie bracket** of X & Y is a vector field $[X,Y]$ defined by

$$[X,Y](f) := X(Yf) - Y(Xf)$$

In coords., if $X = \sum a_i \frac{\partial}{\partial x_i}$ & $Y = \sum b_i \frac{\partial}{\partial x_i}$, then

$$[X,Y]f = X(Yf) - Y(Xf) = X\left(\sum b_i \frac{\partial f}{\partial x_i}\right) - Y\left(\sum a_i \frac{\partial f}{\partial x_i}\right) = \sum_{i,j} a_j \left(\frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} + b_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right) - \sum_{i,j} b_j \left(\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} - a_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right)$$

$$= \sum \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i} + \sum a_i b_j \underbrace{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)}_{0}$$

$$= \underbrace{\left(\sum_{i,j} a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}}_{\text{local coord expression for } [X,Y]} f$$

Prop: If X, Y, Z are smooth vector fields on M & $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

1. $[X,Y] = -[Y,X]$ (anticommutativity)
2. $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ (linearity)
3. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity)
4. $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$