Math 676: Day 29

We've now seen the space $\mathrm{Pol}(n)$ of equilateral polygons in $\mathbb{R}^3$ up to translation & rotation appear as both a

- symplectic reduction $(S^2)^n /\!\!/_{\vec{0}}\, SO(3)$ and as a
- GIT quotient $(\mathbb{CP}^1)^n /\!\!/ PSL(2,\mathbb{C})$.

In fact, this is no coincidence:

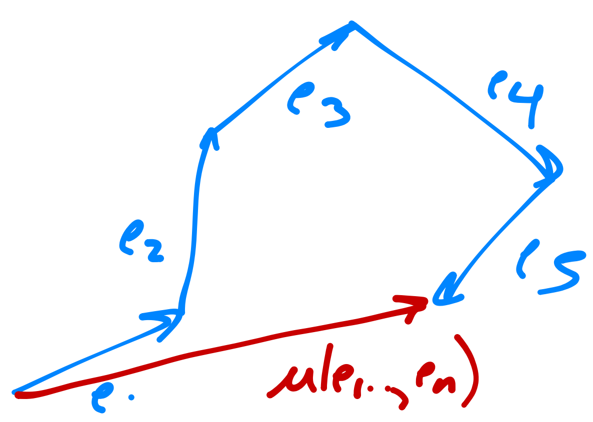
Thm (Kirwan, Kempf-Ness): Suppose G is a reductive algebraic group acting on a projective variety X which is also a Kähler mfd (i.e., X is Riemannian, symplectic & complex-analytic w/ compatibility $g(u,v) = g(Ju,Jv) = \omega(u,Jv)$), that the action of G has a linearization to the tautological line bundle $\mathcal{O}_X(-1)$ over X, & that $K < G$ is the maximal compact subgroup. Then K acts in a Hamiltonian way on X w/ moment map $\mu : X \to \mathfrak{k}^*$ &

$$\underbrace{X /\!\!/_{\vec{0}}\, K}_{\text{symplectic reduction}} = \mu^{-1}(\vec{0})/K \underset{\text{diffeo.}}{\cong} X_{nsst}/G = \underbrace{X /\!\!/ G}_{\text{GIT quotient}}$$

So then the only observation needed is that $SO(3) < PSL(2,\mathbb{C})$ is the maximal compact subgroup (equivalently, $PSL(2,\mathbb{C}) = SO(3)^{\mathbb{C}}$, the complexification of $SO(3)$).

Prop. (Kirwan, Neeman, Sjamaar): With the same setup as in the previous thm, let $F_t(x)$ be the gradient flow of $\|\mu\|^2$ thru $x \in X$ & define $F_\infty(x) = \lim_{t\to\infty} F_t(x)$. Then $F_\infty|_{X_{sst}}$ is a continuous retraction of X_{sst} onto $\mu^{-1}(\vec{0})$.

In the case of polygons, what does this mean? Well, $\mu : (S^2)^n \to \mathbb{R}^3$ is just $\mu(e_1, \ldots, e_n) = \sum e_i$, so $\|\mu\|^2$ is just the squared norm of the failure-to-close vector...



So the point is that all of those fancy constructions turn a random walk into the **same** closed polygon you would get by just doing a gradient descent on the length of the failure-to-close vector, which is surely the most obvious thing one could think of to get from $(S^2)^n$ to $\mathrm{Pol}(n)$.

Okay, so we've seen $Pol(n)$ as the symplectic reduction/GIT quotient of $(S^2)^n = (\mathbb{CP}^1)^n$.

In turn, it's slightly easier to see $(S^2)^n = (\mathbb{CP}^1)^n$ as a symplectic reduction/GIT quotient of $\mathbb{C}^{n\times 2} = Mat_{n\times 2}(\mathbb{C})$.

We're going to consider the obvious torus action on $\mathbb{C}^{n\times 2}$: algebraically, we take $(\mathbb{C}^*)^n$ to be the diagonal matrices in $GL(n,\mathbb{C})$;

symplectically, we take $(S^1)^n = (U(1))^n$ to be the diagonal unitary matrices.

Recall that we showed that the diagonal $U(1)$ action on $\mathbb{C}^m$ has moment map $\vec{z} \mapsto -\frac{1}{2}|\vec{z}|^2 + \frac{1}{2}$. Hence, repeatedly using $m=2$, the $U(1)^n$ action on $\mathbb{C}^{n\times 2}$ has moment map $\mu: \mathbb{C}^{n\times 2} \to \mathbb{R}^n$

$$\mu\left(\begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2}|(a_1,b_1)|^2 + \frac{1}{2} \\ \vdots \\ -\frac{1}{2}|(a_n,b_n)|^2 + \frac{1}{2} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} a_1\bar{a}_1 + b_1\bar{b}_1 - 1 \\ \vdots \\ a_n\bar{a}_n + b_n\bar{b}_n - 1 \end{bmatrix}$$

So then $\mu^{-1}(\vec{0}) = \left\{ \begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} : a_1\bar{a}_1 + b_1\bar{b}_1 = 1 \right\} = S^3(1) \times \ldots \times S^3(1)$.

(By choosing potentially different constants in each factor, can get 3-spheres of any radii).

But then the symplectic reduction is $\mathbb{C}^{n\times 2} /\!\!/_{\vec{0}} (U(1))^n = \mu^{-1}(\vec{0})/(U(1))^n = \overbrace{S^3(1) \times \ldots \times S^3(1)}^{n} / (U(1))^n$

But now the orbits of each $U(1)$ are just the intersection of the corresponding S^3 w/ all the complex lines in $\mathbb{C}^2$...

so the quotient is just $\underbrace{\mathbb{CP}^1 \times \ldots \times \mathbb{CP}^1}_{n}$. Of course, the same is true for the GIT quotient, so we get

the following chain of iterated symplectic reductions/GIT quotients

$$\begin{array}{l} \mathbb{C}^{n\times 2} \\ \Big| \; /\!\!/_{\vec{0}} (U(1))^n \text{ or } /\!\!/ (\mathbb{C}^*)^n \\ (S^2)^n \cong (\mathbb{CP}^1)^n \\ \Big| \; /\!\!/_{\vec{0}} SO(3) \text{ or } /\!\!/ PSL(2,\mathbb{C}) \\ Pol(n) \end{array}$$