Math 676: Day 26

So far, we understand $(\mathbb{P}^1)^n /\!/ PSL(2,\mathbb{C})$ pretty well: it's just the geometric quotient of nice semistable points by the action of $PSL(2,\mathbb{C})$.

But in what sense does this correspond to the collection of **closed** polygons, as claimed?

In this setting, it's natural to think of S^2 as the boundary of the unit ball, which we identify with the Poincaré ball model of hyperbolic 3-space $\mathbb{H}^3$, & $PSL(2,\mathbb{C})$ is the isometry group of $\mathbb{H}^3$.

Now, we use a construction from Douady–Earle:

Df: A probability measure ν on S^2 is **stable** if $\nu(p) < 1/2$ for any $p \in S^2$, & it is **semi-stable** if $\nu(p) \leq 1/2$ $\forall$ $p \in S^2$. A measure is **nice semi-stable** if it has exactly 2 atoms of mass $1/2$.

Ex: Given a weighted collection of points $((e_1,\dots,e_n),(r_1,\dots,r_n))$ (recall $e_i \in S^2$ & $\sum r_i = 2$), we get a probability measure

$$\nu = \frac{1}{2}\sum r_i \delta_{e_i},$$

which is stable/semi-stable/nice semi-stable $\Leftrightarrow$ the weighted points are according to the earlier definition.

Now, if $i: S^2 \hookrightarrow \mathbb{R}^3$ is the inclusion, then the center of mass $B(\nu)$ of a measure ν on S^2 is defined by

$$B(\nu) = \int_{S^2} i(p)\, d\nu(p)$$

For measures of the form $\nu = \frac{1}{2}\sum r_i \delta_{e_i}$ we have

$$B(\nu) = \int_{S^2} i(p)\, d\nu(p) = \int_{S^2} p \cdot \frac{1}{2}\sum r_i \delta_{e_i}(p)\, dArea = \sum \frac{r_i}{2} e_i.$$

For example, when $(r_1,\dots,r_n) = (\frac{2}{n},\dots,\frac{2}{n})$, this is just $\sum \frac{1}{n} e_i = \frac{1}{n}\sum e_i$.

In general, if ν is an atomic measure, then $B(\nu)$ is the usual weighted center of mass of the atoms.

Prop: For each stable measure ν on S^2 $\exists$ $\gamma \in PSL(2,\mathbb{C})$ s.t. $B(\gamma_* \nu) = 0$. The element γ is unique up to postcomposition by $g \in SO(3) < PSL(2,\mathbb{C})$.

Translating, this means each $PSL(2,\mathbb{C})$ orbit in M_{st} contains a unique closed polygon up to rotation.

Since we already saw that each equivalence class in Q_{cusp} has a unique $PSL(2,\mathbb{C})$ orbit of nice semi-stable points,

& since each such orbit contains a unique (up to rotation) closed polygon, we see that points in

$$(\mathbb{P}^1)^n /\!/ PSL(2,\mathbb{C}) \cong M_{nsst} / PSL(2,\mathbb{C})$$

can be uniquely represented by a closed n-gon up to rotation... in other words, that

$$((S^2)^n, \omega) /\!/_{\vec{0}}\, SO(3) = \mu^{-1}(\vec{0})/SO(3) \cong M_{nsst}/PSL(2,\mathbb{C}) \cong (\mathbb{P}^1)^n /\!/ PSL(2,\mathbb{C}).$$