Math 676: Day 25

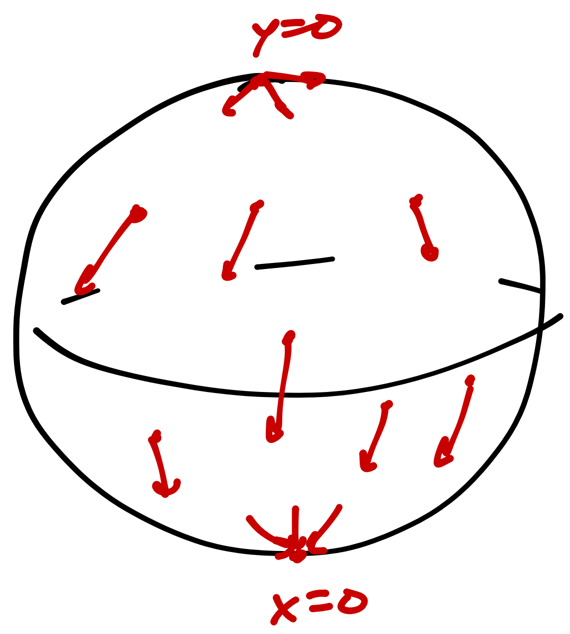
Recall from last time the characterization of semistable points on $(\mathbb{P}^1)^n$ under the $PSL(2,\mathbb{C})$ action:

Prop: ① $(p_1,\dots,p_n)\in(\mathbb{P}^1)^n$ is semistable $\iff$ no more than $n/2$ of the p_i's are equal.

② $(p_1,\dots,p_n)\in(\mathbb{P}^1)^n$ is stable $\iff$ fewer than $n/2$ of the p_i's are equal.

Pf: Exercise.

Geometrically, the points on $\mathbb{P}^1$ flow towards some point (say $x=0$) unless they're stuck at the antipodal point ($y=0$).



We can more generally think of *weighted* configurations of n points on $S^2\cong\mathbb{P}^1$: this is a collection

$(e_1,\dots,e_n)\in S^2$ together w/ weights $(r_1,\dots,r_n)\in(\mathbb{R}_+)^n$, which we will normalize so that

$r_1+\dots+r_n=2$. Think of the weights as edgelengths, so we get a corresponding random walk w/ edge vectors

$(r_1e_1,\dots,r_ne_n)$. The equilateral case we've been considering is (up to scale) when $r_i=\frac{2}{n}$ for all $i=1,\dots,n$.

Now, we follow Deligne–Mostow's construction of the weighted quotient of n points on S^2, which is just a concrete version of $(\mathbb{P}^1)^n/\!/PSL(2,\mathbb{C})$, where choosing rational $(r_1,\dots,r_n)$ corresponds to the line bundle

$$L_{\vec r}=\boxtimes_i \mathcal{O}(2Dr_i),$$ where D is the common denominator of the r's.

Def: Given weights $\vec r=(r_1,\dots,r_n)$, the point $(e_1,\dots,e_n)\in(S^2)^n$ is called *$\vec r$-stable* (respectively, *$\vec r$-semistable*) if

$$\sum_{e_i=v} r_i<1 \quad \left(\text{respectively, } \sum_{e_i=v} r_i\le 1\right) \text{ for all } v\in S^2.$$

Let $M_{st}\subseteq(S^2)^n$ be the set of stable points, & let $M_{sst}\subseteq(S^2)^n$ be the semistable points.

A semistable point is *nice semistable* if it is either stable or its $PSL(2,\mathbb{C})$ orbit is closed in M_{sst}; we use M_{nsst} for the set of nice semistable points.

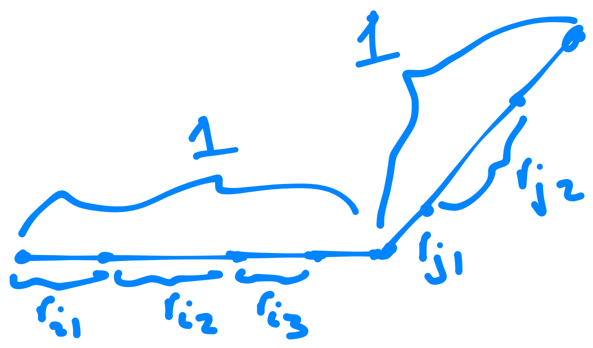
In polygon terms, the stable points correspond to configurations which could be made into nondegenerate polygons after applying some Möbius transformation, the semistable points are those that could conceivably be polygons, but only by being lined. The unstable points are those which could never be made closed.

Let $M_{cusp} = M_{sst} \setminus M_{st}$. Points in M_{cusp} correspond to partitions $P_1 \sqcup P_2$ of $\{1, \dots, n\}$ s.t.

$S_1 = \{i_1, \dots, i_k\}$, $S_2 = \{j_1, \dots, j_{n-k}\}$ & $r_{i_1} + \dots + r_{i_k} = 1$ (& hence $r_{j_1} + \dots + r_{j_{n-k}} = 1$):

$(e_1, \dots, e_n) \in M_{cusp} \iff \exists$ such a partition s.t. either $e_{i_1} = \dots = e_{i_k}$ or $e_{j_1} = \dots = e_{j_{n-k}}$,

and a point is in $M_{nsst} \cap M_{cusp} \iff$ both of the above equalities hold, meaning the corresponding random walk looks like (up to permutation of the edge order)



For equilateral walks, this just means half the edges point in one direction & half point in the other.

Now, $(\mathbb{P}^1)^n /\!/ PSL(2,\mathbb{C}) \cong M_{sst}/\sim$, where $\sim$ is the equivalence relation given by:

$(e_1, \dots, e_n) \sim (f_1, \dots, f_n)$ if either

① $(e_1, \dots, e_n), (f_1, \dots, f_n) \in M_{st}$ & they are in the same $PSL(2,\mathbb{C})$ orbit

or ② $(e_1, \dots, e_n), (f_1, \dots, f_n) \in M_{cusp}$ & they correspond to the same partition of $\{1, \dots, n\}$.

Notice that if $(e_1, \dots, e_n) \sim (f_1, \dots, f_n)$ & $(e_1, \dots, e_n), (f_1, \dots, f_n) \in M_{nsst}$, then they are in the same $PSL(2,\mathbb{C})$ orbit.

Let $Q_{sst} := M_{sst}/\sim$, $Q_{nsst} := M_{nsst}/\sim$, $Q_{st} := M_{st}/\sim$, $Q_{cusp} := M_{cusp}/\sim$.

Since elements of Q_{cusp} are uniquely determined by their partitions, Q_{cusp} is a finite set of points. Moreover, each equivalence class in Q_{cusp} contains a unique $PSL(2,\mathbb{C})$-orbit of nice semistable points. Hence,

$$Q_{nsst} = M_{nsst}/PSL(2,\mathbb{C}) \cong Q_{sst} = (\mathbb{P}^1)^n / PSL(2,\mathbb{C})$$