Math 676: Day 24

Recall, we're talking about the GIT quotient $X /\!/ G$, where G is an algebraic gp acting on the projective variety X

(&, implicitly, on the tautological line bundle $\mathcal{O}_X(-1)$ on X).

Def: $x \in X$ is **semistable** if $\exists$ $s \in H^0(X, \mathcal{O}_X(r))^G$ with $r > 0$ s.t. $s(x) \neq 0$.

Def: A semistable point $x \in X$ is **stable** if $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ separates orbits near x & the stabilizer of x is finite.

Now, what's going on geometrically?

Thm: Suppose $x \in X$ & $\tilde{x} \in \mathcal{O}_X(-1)$ covers x. Then

① x is semistable $\iff$ $0 \notin \overline{G.\tilde{x}}$

② x is stable $\iff$ $G.\tilde{x}$ is closed in $\mathbb{C}^{n+1}$ & $\tilde{x}$ has finite stabilizer.

Idea: G-invt homogeneous functions of degree $r > 0$ on $\tilde{X}$ are constant on orbits, & hence on their closures. So if

$\overline{G.\tilde{x}} \ni 0$, then all such functions are zero & so x is not semistable.

Moreover, if x is stable, then invt functions separate orbits around $G.\tilde{x}$, so $G.\tilde{x}$ is the zero locus of some

collection of invt. functions, & hence closed.

Prop: $x \in X_{sst}$ is stable $\iff$ the orbit $G.x$ is closed in X_{sst} & $\dim G.x = \dim G$.

Of course, understanding entire group orbits is generally hard... it's typically easier to understand **1-parameter subgroups**, i.e., copies

of $\mathbb{C}^* < G$.

We can apply the theorem above to the orbits of all 1-parameter subgroups, where we have the following easier characterization:

For $x \in X$, let $x_0 = \lim_{\lambda \to 0} \lambda.x$, which is a fixed point of the $\mathbb{C}^*$ action, so $\mathbb{C}^*$ acts on the line l_{x_0} in $\mathbb{C}^{n+1}$

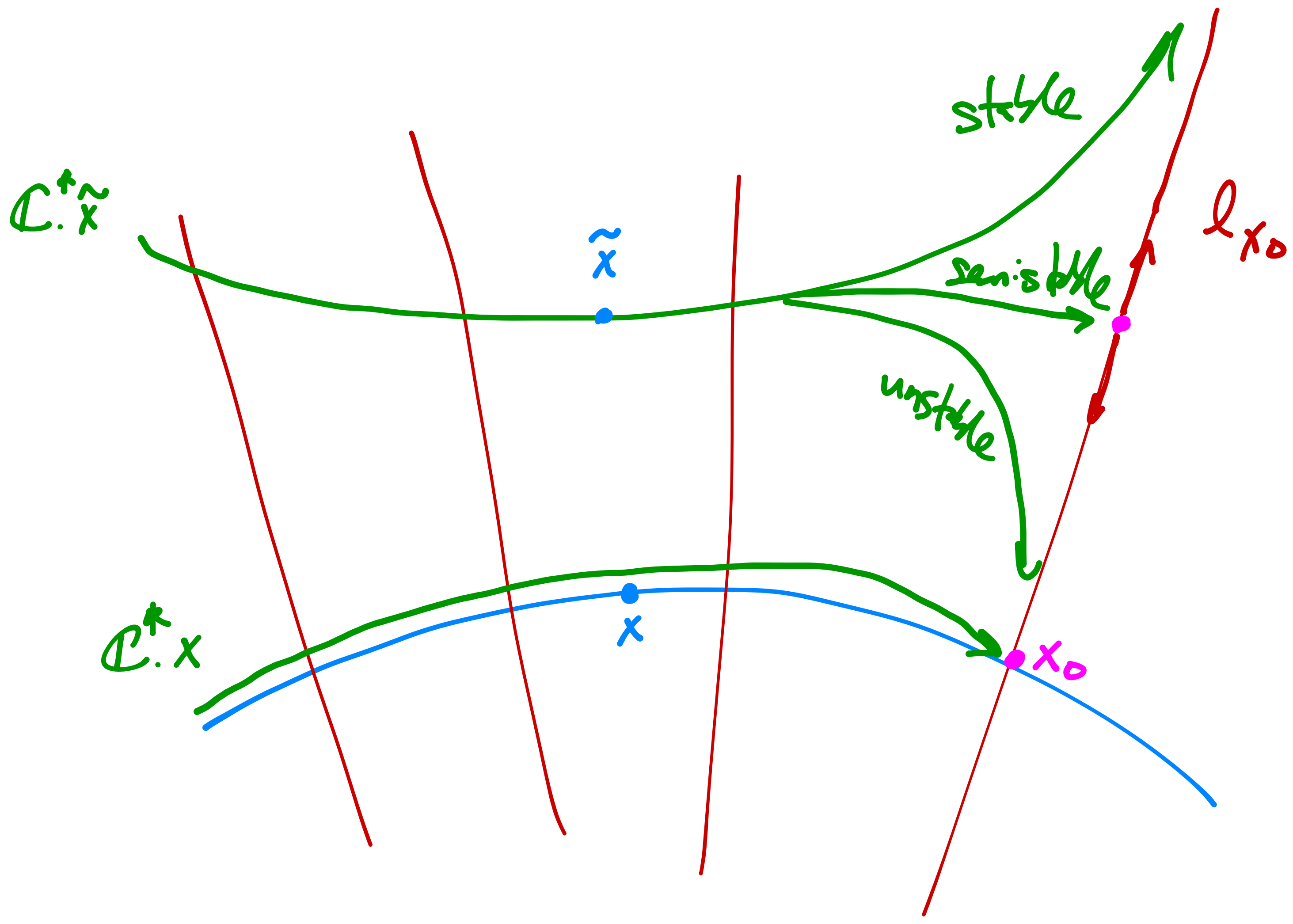
represented by $x_0 \in \mathbb{P}^n$. Of course, a $\mathbb{C}^*$ action on a line is of the form $\lambda . z = \lambda^\rho z$ where ρ is the

weight of the action. So define $\rho(x)$ to be the weight of the induced $\mathbb{C}^*$ action on l_{x_0}.

Hilbert-Mumford Criterion:

- If $\rho(x) < 0$ for all 1-parameter subgps, then x is stable.
- If $\rho(x) \leq 0$ for all 1-parameter subgps, then x is semistable
- If $\rho(x) > 0$ for some 1-parameter subgp, then x is unstable.

Idea:



The orbit is closed $\Leftrightarrow$ it is asymptotic to a negative weight $\mathbb{C}^*$-action on the limiting line at both $\lambda \to 0$ & $\lambda \to \infty$.

But it suffices to consider $\lambda \to 0$, since the $\lambda \to \infty$ case is the $\lambda \to 0$ case of the inverse 1-param subgp.

Ex: Now, we consider n points on $\mathbb{P}^1$ & the diagonal action of $PGL(2, \mathbb{C}) = PSL(2, \mathbb{C})$.

Specifying n points is the same as specifying a degree n homogeneous poly. whose roots are the points

(strictly speaking, this only specifies n unordered points, or a point in the symmetric product $\Sigma^n \mathbb{P}^1$ rather than in $(\mathbb{P}^1)^n$, but I'm going to ignore this)

Prop: ① $(p_1, \ldots, p_n) \in (\mathbb{P}^1)^n$ is semistable $\Leftrightarrow$ no more than $n/2$ of the p_i's are equal.

② $(p_1, \ldots, p_n) \in (\mathbb{P}^1)^n$ is stable $\Leftrightarrow$ fewer than $n/2$ of the p_i's are equal.

Pf: Exercise.