Math 676: Day 23

Last time, we said that if X is a projective variety & G is an (reductive, complex linear) algebraic group acting on X through isomorphisms of the ambient projective space:

$$\begin{tikzcd} G \arrow[d] \arrow[r, phantom, "\curvearrowright"] & X \arrow[d, phantom, "\cap"] \\ SL(n+1,\mathbb{C}) \arrow[r, phantom, "\curvearrowright"] & \mathbb{P}^n \end{tikzcd}$$

Then the GIT quotient $X/\!/G$ is given by $\mathrm{Proj} \bigoplus_r H^0(X, \mathcal{O}(r))^G$.

Let's break this down a bit: for a graded ring R, $\mathrm{Proj}\, R$ is the projective variety whose points are the homogeneous ideals of R which are maximal among those not including $R^+ := \bigoplus_{k>0} R_k$. This comes w/ an ample line bundle, which we think of as $\mathcal{O}(1)$ for the variety.

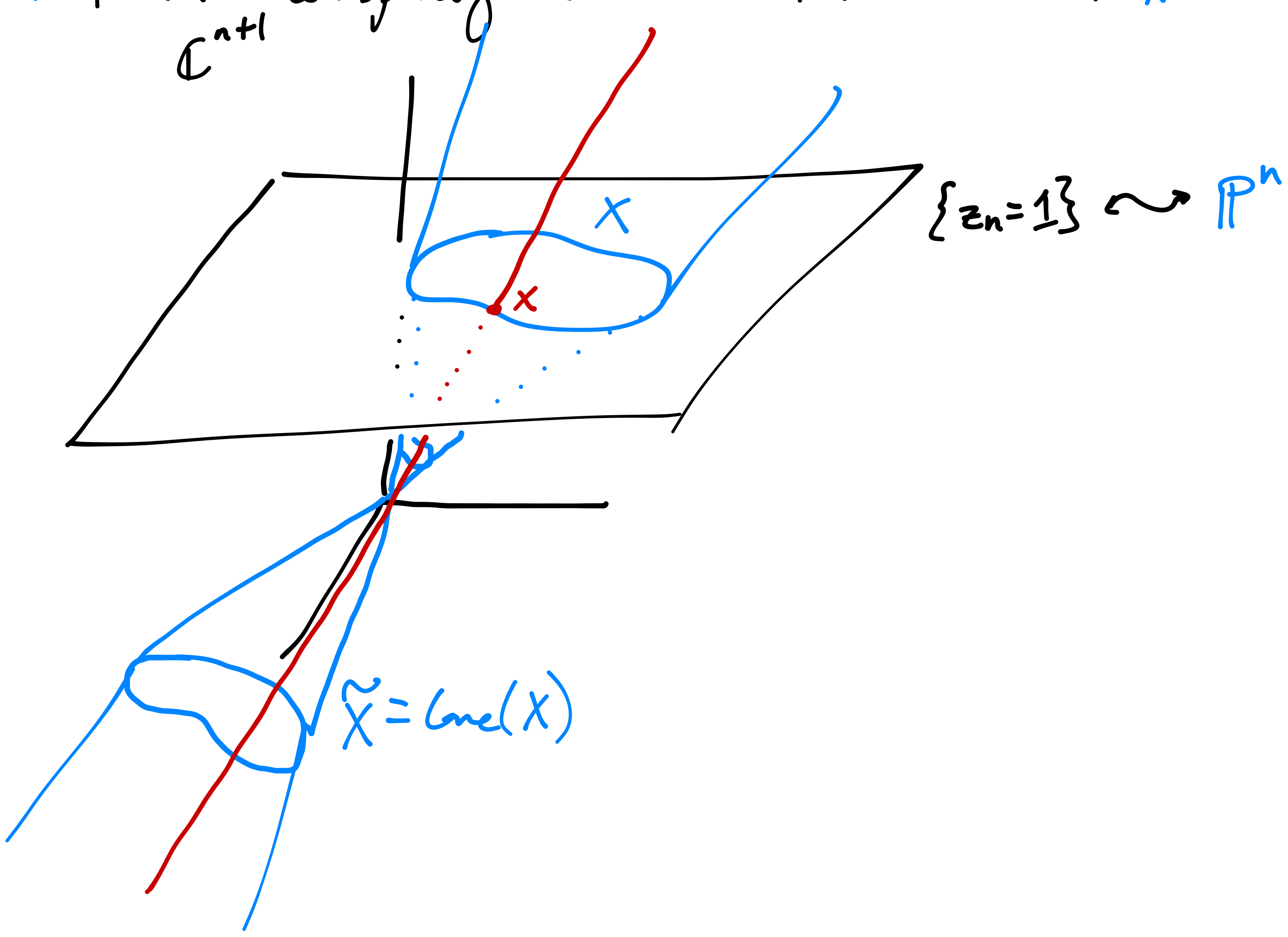
Also, $H^0(X, \mathcal{O}(r))$ is the sections of the bundle $\begin{matrix}\mathcal{O}(r)\\ \downarrow \\ X\end{matrix}$ which is the r^{th} tensor power of the dual $\mathcal{O}_X(1)$ of the tautological line bundle $\mathcal{O}_X(-1)$, and $H^0(X, \mathcal{O}(r))^G$ are the G-invariant sections.

Now, the point is that $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ is supposed to play the role of the ring of functions on $X/\!/G$... so why is that?

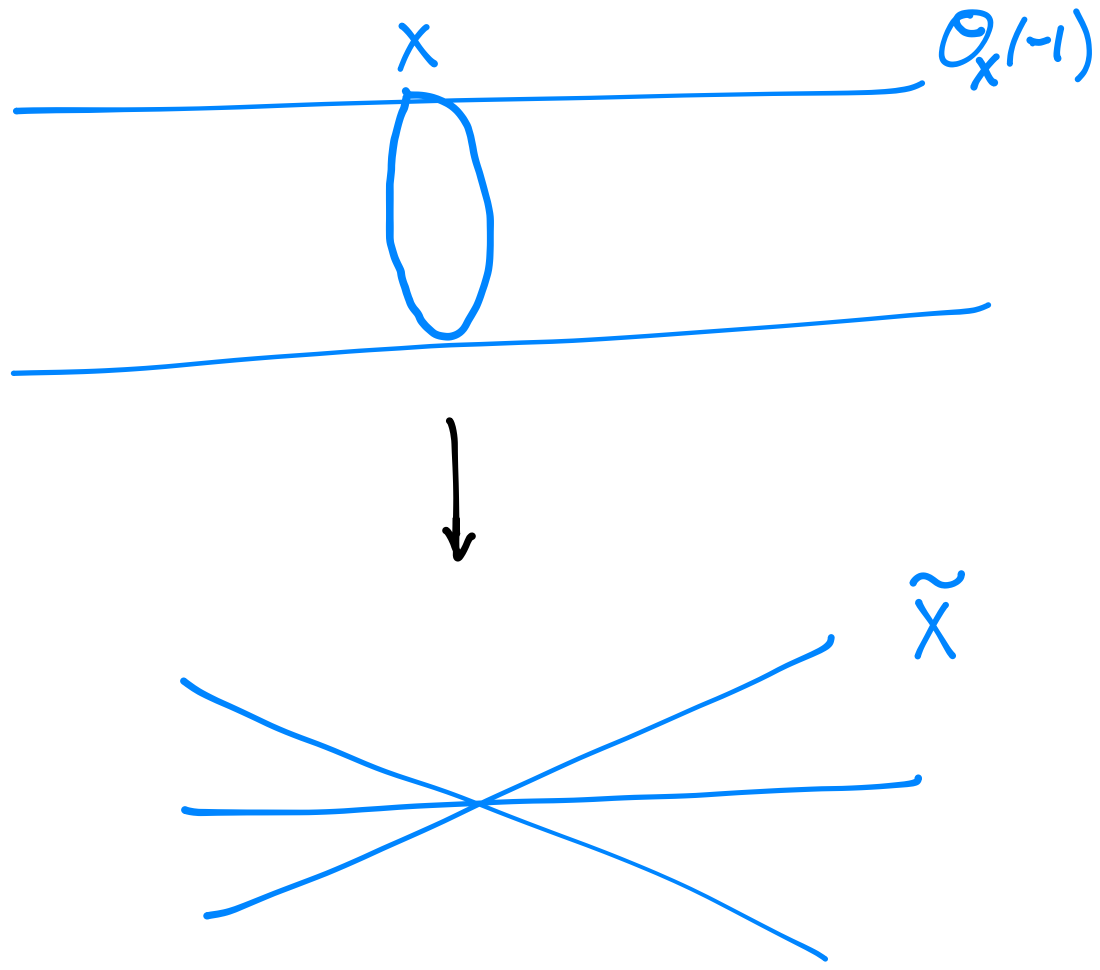
And also, where did $\mathcal{O}_X(-1)$ come from, anyway?

Well, $X \subset \mathbb{P}^n$, so points in X are lines in $\mathbb{C}^{n+1}$, & so we get the tautological line bundle $\mathcal{O}_X(-1)$ by taking the fiber over each point $x \in X$ to be the corresponding line in $\mathbb{C}^{n+1}$. In other words, $\mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}^n}(-1)|_X$.

Pictorially:



Of course $\mathcal{O}_X(-1)$ is almost $\tilde{X}$, except that the zero sections are diff't. In fact, $\mathcal{O}_X(-1)$ is the blowup at $\vec{0}$ of $\tilde{X}$



So linear functions on $\mathbb{C}^{n+1}$ restricted to $\tilde{X}$ & then pulled back to $\mathcal{O}_X(-1)$ give functions which are linear on the fibers,

meaning they are equivalent to sections of the dual bundle $\begin{array}{c}\mathcal{O}_X(1)\\ \downarrow \\ X\end{array}$

More generally, degree k homogeneous polynomials on $\mathbb{C}^{n+1}$ give sections of the k^{th} tensor power $\mathcal{O}_X(k)$.

Of course, all together $\bigoplus_k H^0(X, \mathcal{O}_X(k))$ gives the graded (by degree) ring of homogeneous functions on the cone $\tilde{X}$

which is the right notion of the ring of functions on the projective variety X.

In turn, the G-invariant sections should give the "ring of functions" on the quotient $X/\!/G$, & applying Proj then recovers the variety w/ line bundle.

In general, then, this construction really applies to any projective variety w/ an (ample) line bundle & a group action which extends to the line bundle (i.e., has a linearization).

Now, this doesn't really tell us what the points of $X/\!/G$ actually are. To do so, we need to define (semi)stable points on X.

Recall that $X/\!/G = \mathrm{Proj} \bigoplus_r H^0(X, \mathcal{O}_X(r))^G$. The relevant questions are:

① What points of X can $\bigoplus_r H^0(X, \mathcal{O}_X(r))^G$ even see?

② What points that it can see can't be distinguished?

More precisely, $X/\!\!/G$ can be realized concretely as the image of

$$X \dashrightarrow \mathbb{P}\left(\left(H^0(X,\mathcal{O}(r))^G\right)^*\right)$$
$$x \longmapsto ev_x \quad \text{where } ev_x(s) := s(x),$$

for $r \gg 0$. In coords., if $s_0, \ldots, s_k$ is a basis for $H^0(X,\mathcal{O}(r))^G$, then $x \mapsto [s_0(x) : s_1(x) : \cdots : s_k(x)]$, which of course only makes sense if $s_i(x)$ are not all zero. In other words, this map is defined on the locus of semistable points:

Def: $x \in X$ is semistable if $\exists\ s \in H^0(X, \mathcal{O}_X(r))^G$ with $r > 0$ s.t. $s(x) \neq 0$.

So if X_{ss} is the set of semistable points, then the above map is really only defined on X_{sst}.

Since the map is constant on G-orbits, it factors thru X_{sst}/G ... but may not be equal to X_{sst}/G.

Def: A semistable point $x \in X$ is stable if $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ separates orbits near x & the stabilizer of x is finite.

Clearly, on the stable points X_{st}, the GIT quotient is just the usual quotient: $X_{st}/\!\!/G = X_{st}/G$.