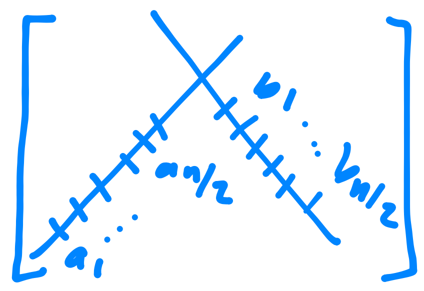
Math 676: Day 22

Last time, we saw that the group $SO(3)$ does not act freely on the closed polygons $\mu^{-1}(\vec{0}) \subseteq (S^2)^n$ when n is even: the lined polygons are fixed points of a copy of $SO(2) \leq SO(3)$.

In $\overline{M}_{0,n}$ language, these correspond to the boundary divisor $D(\{a_1,\dots,a_{n/2}\} \mid \{b_1,\dots,b_{n/2}\}) =$ 

However, if we delete the preimages of these points from $(S^2)^n$ (& hence from $\mu^{-1}(\vec{0})$), then the Marsden-Meyer-Weinstein theorem applies & we get that the closed polygons (except the degenerate ones) up to translation & rotation are a nice symplectic mfd.

In fact, putting the bad points back in isn't too bad:

Thm (Kapovich-Millson '96): The space of equilateral n-gons up to translation & rotation, a.k.a. $(S^2)^n /\!\!/_0 SO(3)$, is a complex analytic space. It has isolated singularities corresponding to the degenerate n-gons, & these singularities are equivalent to homogeneous quadratic cones.

specifically, of the form $\{(z,w) \in \mathbb{C}^{n/2} \times \mathbb{C}^{n/2} : \text{either } z \neq 0 \;\&\; w \neq 0, \text{ or } (z,w) = (0,0)\} / \mathbb{C}^*$

where the $\mathbb{C}^*$ action on $\mathbb{C}^n = \mathbb{C}^{n/2} \times \mathbb{C}^{n/2}$ is given by

$$\lambda \cdot (z,w) = (\lambda z, \lambda^{-1} w).$$

This is an affine variety corresponding to the ring of invariants $\mathbb{C}[z_1,\dots,z_{n/2},w_1,\dots,w_{n/2}]^{\mathbb{C}^*}$, which is generated by $f_{ij} = z_i w_j$, w/ relations $f_{ij} f_{ji} = f_{ii} f_{jj}$, & so the quotient is a homogeneous quadratic cone.

Kapovich-Millson proved this by realizing that $(S^2)^n /\!\!/_0 SO(3)$ is diffeomorphic to the GIT quotient of $(\mathbb{P}^1)^n$ by $PGL(2,\mathbb{C})$, which I now want to describe in differential geometry language.

This is closely connected to $\overline{M}_{0,n}$: in fact, this GIT quotient is a different choice of compactification of $M_{0,n}$.

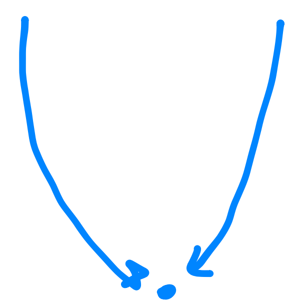
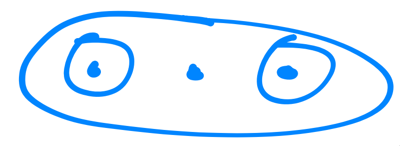
Of course, $PGL(2,\mathbb{C})$ is the automorphism group of $\mathbb{P}^1$, & so

$$\left((\mathbb{P}^1)^n \setminus \Delta\right)/PGL(2,\mathbb{C})$$

is exactly the moduli space of n distinct points on $\mathbb{P}^1$.

Now, in Renzo's class we've seen a way of compactifying this into $\overline{M}_{0,n}$ which depends fairly strongly on the particular geometry of pointed curves.

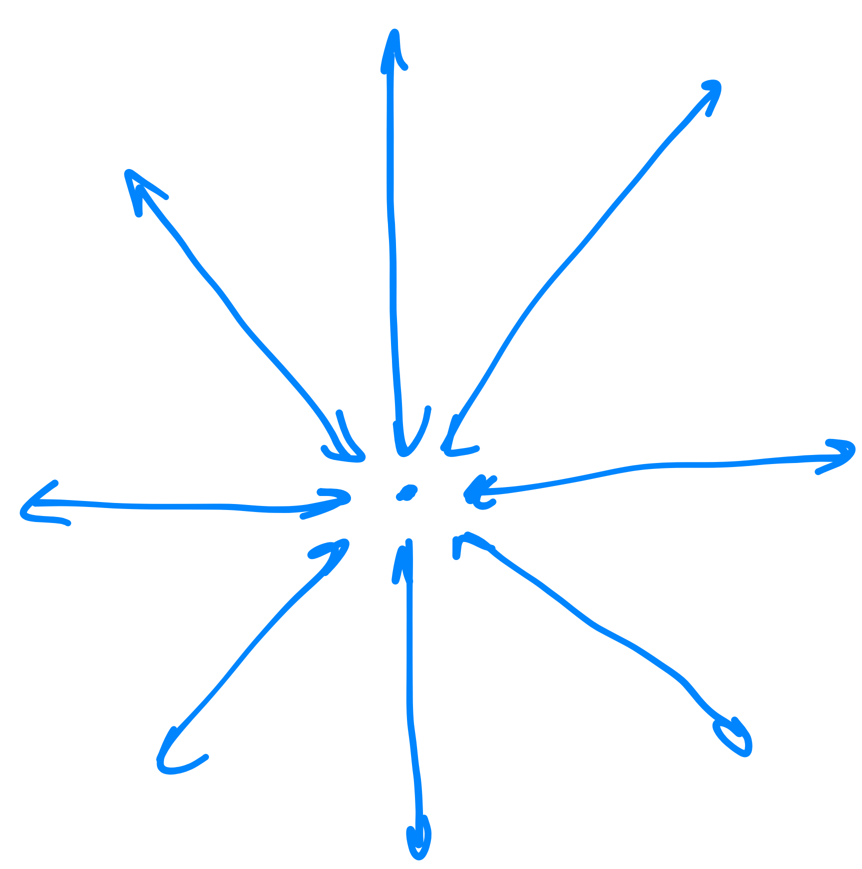
More generally, though, the point is that $PGL(2,\mathbb{C})$ acts on $(\mathbb{P}^1)^n$, but that, since $(\mathbb{P}^1)^n$ is compact & $PGL(2,\mathbb{C})$ is not, the action cannot be proper: $\exists$ nonclosed orbits.

Schematic picture: Think of an action w/ 3 orbits:  All three closures contain the special point, so the topological quotient is , which is non-Hausdorff.

In this case, can fix by deleting the lower-dim'l orbit.

Notice that this is the same thing we do when we look at orbits in $\mathbb{C}^{n+1}$ of the natural $\mathbb{C}^*$ action:

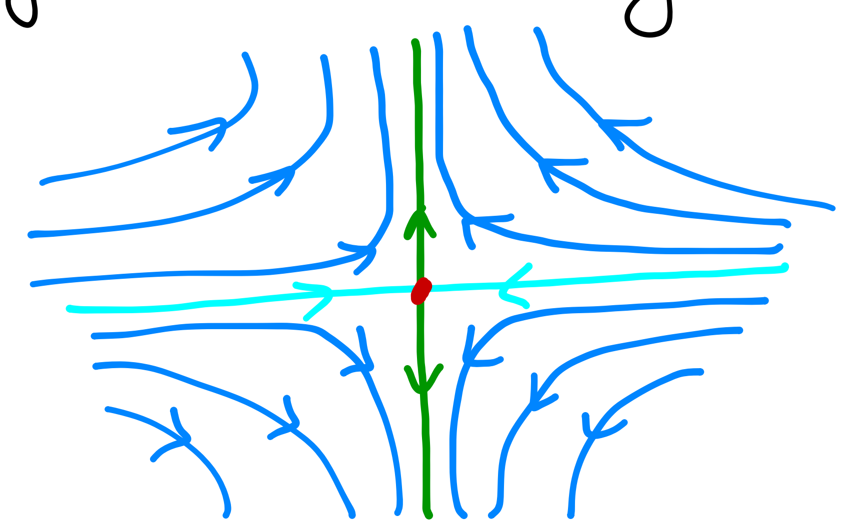
orbits look like



All orbits have the origin in their closure, so we instead consider $(\mathbb{C}^{n+1}\setminus\{\vec{0}\})/\mathbb{C}^* = \mathbb{CP}^n$.

Ex: Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ (or $\mathbb{CP}^2$) by $\mathbb{C}^* \ni \lambda \longleftrightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in SL(2,\mathbb{C})$ (or $PSL(2,\mathbb{C}) = PGL(2,\mathbb{C})$)

orbits:



Generic orbits are of the form $\{xy=\alpha\}$ for $\alpha \neq 0$.

The case $\alpha=0$ breaks down into 3 special orbits:
- $\{(0,y) : y\neq 0\}$
- $\{(x,0) : x \neq 0\}$
- $\{(0,0)\}$

Now, even if we delete $(0,0)$, the quotient is the α-line, but w/ a nonseparated double point at the origin since the limit as $\alpha \to 0$ of a generic orbit is $\{x=0\} \cup \{y=0\}$.

Of course, we want the quotient to be $\mathbb{C}$, but what to do w/ $\alpha=0$?

3 answers (following Thomas):

① Chow quotient: represent $\alpha=0$ by $\{xy=0\}$, the union of all 3 bad orbits.

② GIT quotient: Identify all 3 bad orbits since their orbits have intersecting closures

③ Symplectic reduction: throw away the nonclosed orbits

Formally, if X is projective & the G-action extends to an $SL(n+1,\mathbb{C})$ action on $\mathbb{P}^n \supseteq X$, then define $X/\!/G := \mathrm{Proj} \bigoplus_r H^0(X, \mathcal{O}(r))^G$

In the affine case, if R is the ring of functions on X, then guess that $X/\!/G = \mathrm{Spec}(R^G)$.

Indeed, this totally works in the example: $R = \mathbb{C}[x,y]$ & $R^G = \mathbb{C}[x,y]^{\mathbb{C}^*} = \mathbb{C}[xy]$ & $\mathrm{Spec}(\mathbb{C}[xy]) = \mathbb{C}$.

But in other cases this is bad: consider the usual $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$. The only invariant polys. in $\mathbb{C}[x_0,..,x_n]$ are the constants, so $\mathrm{Spec}(R^G) = \{\cdot\}$, & we don't really want to think of $\mathbb{C}^{n+1}/\!/\mathbb{C}^*$ as being a single point

(of course, the fix here is to use the above definition)