Math 676: Day 21

Recall:

Thm (Marsden-Weinstein-Meyer): Suppose the cpct Lie group G acts in a Hamiltonian way on the symplectic mfd (M,ω), w/ moment map $\mu: M \to \mathfrak{g}^*$. Let $i: \mu^{-1}(0) \hookrightarrow M$ be the inclusion & suppose G acts freely on $\mu^{-1}(0)$. Then

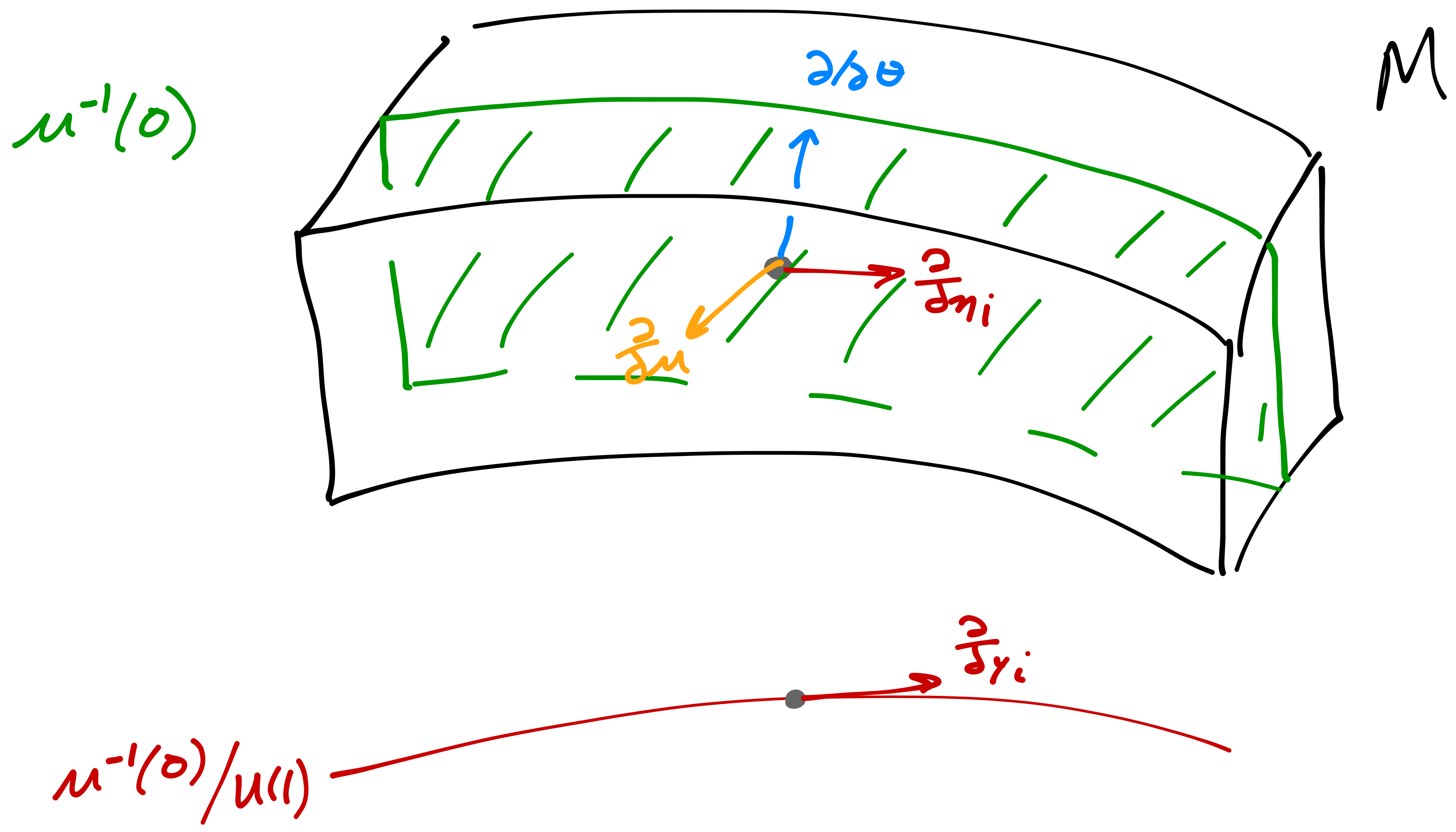
1. The orbit space $M/\!/_0 G := \mu^{-1}(0)/G$ is a mfd
2. $\pi: \mu^{-1}(0) \to M/\!/_0 G$ is a principal G-bundle (i.e. $\mu^{-1}(0)$ is locally a product)
3. There is a symplectic form ω_{red} on $M/\!/_0 G$ s.t. $i^*\omega = \pi^*\omega_{red}$

Now, the first nontrivial case where the theorem says something is when $G = S^1 = U(1)$ & $\dim M = 4$.

We can give a low-tech proof in this case:

Pf (for $\dim M = 4$, $G = U(1)$): In this case, $\mathfrak{g}^* \cong \mathbb{R}$, so $\mu: M \to \mathbb{R}$. Let $p \in \mu^{-1}(0)$ & choose local coords. in a nbhd of p:

- θ along the orbit of p
- μ given by the moment map
- η_1, η_2 pullbacks of local coords y_1, y_2 around $\pi(p) \in \mu^{-1}(0)/U(1)$.



Then the symplectic form ω is a 2-form, & so must have the local coord. expression

$$\omega = A\, d\theta \wedge d\mu + \sum_{i=1}^{2} \left(B_i\, d\theta \wedge d\eta_i + C_i\, d\mu \wedge d\eta_i \right) + D\, d\eta_1 \wedge d\eta_2$$

for some functions A, B_1, B_2, C_1, C_2, D.

But now μ is the Hamiltonian function for $\frac{\partial}{\partial\theta}$, so

$$d\mu = \iota_{\frac{\partial}{\partial\theta}}\omega = \iota_{\frac{\partial}{\partial\theta}}\left(A\,d\theta\wedge d\mu + \sum\left(B_i\,d\theta\wedge d\eta_i + C_i\,d\mu\wedge d\eta_i\right) + D\,d\eta_1\wedge d\eta_2\right) = A\,d\mu + \sum_{i=1}^{2} B_i\,d\eta_i,$$

so $A \equiv 1$ & $B_1 \equiv B_2 \equiv 0$.

Next, since ω is symplectic it must be nondegenerate, which is equivalent to $\omega\wedge\omega$ being a volume form.

But $\omega\wedge\omega = \left(d\theta\wedge d\mu + C_1\,d\mu\wedge d\eta_1 + C_2\,d\mu\wedge d\eta_2 + D\,d\eta_1\wedge d\eta_2\right)\wedge\left(A\,d\theta\wedge d\mu + C_1\,d\mu\wedge d\eta_1 + C_2\,d\mu\wedge d\eta_2 + D\,d\eta_1\wedge d\eta_2\right)$

$$= D\,d\theta\wedge d\mu\wedge d\eta_1\wedge d\eta_2,$$

so D is never 0. But now notice $i^*\omega = D\,d\eta_1\wedge d\eta_2 = \pi^*(D\,dy_1\wedge dy_2)$ is nowhere-vanishing, & so

$D\,dy_1\wedge dy_2$ is a nowhere-vanishing 2 form on the surface $\mu^{-1}(0)/U(1)$, & hence a symplectic form. ∎

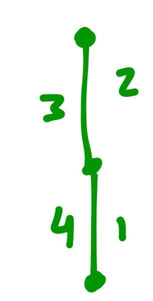
Now, let's see what this theorem tells us about random walks.

Remember that the space of equilateral random walks up to translation is $(M,\omega) = (\underbrace{S^2\times\cdots\times S^2}_{n}, \sum_{i=1}^{n}\pi_i^*\omega_{std})$ & that the rotation group $SO(3)$ acts diagonally. Moreover, we saw that this $SO(3)$ action is Hamiltonian, & that the moment map is $\mu: (S^2)^n \to so(3)^* \cong \mathbb{R}^3$ given by $\mu(\vec{e}_1,\ldots,\vec{e}_n) = \vec{e}_1+\cdots+\vec{e}_n$.

Now, $\mu^{-1}(\vec{0})$ consists of the closed polygons, & so the closed equilateral polygons up to translation & rotation would be

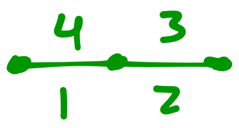
$$\mu^{-1}(\vec{0})/SO(3) = (S^2)^n /\!\!/_{0}\, SO(3)$$

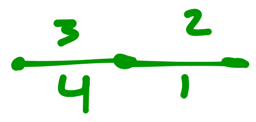
The problem is that $SO(3)$ does not act freely on $\mu^{-1}(\vec{0})$. For example, consider $n=4$ & the point $P = ((1,0,0),(1,0,0),(-1,0,0),(-1,0,0)) \in \mu^{-1}(\vec{0})$, which gives the polygon

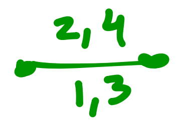


But now, the subgroup $\left\{\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} \,\middle|\, \theta\in\mathbb{R}\right\} \subseteq SO(3)$, which is a copy of $SO(2)$ which rotates around the z-axis, has no effect on P : P is a fixed point of the entire subgroup.

Now, in the quotient $\mu^{-1}(\vec{0})/SO(3)$ there are only 3 bad points: the images of P & the various permutations of its edges, of which there are only 3 distinct types: out-out-back-back (like P), out-back-back-out, & out-back-out-back.







In terms of $\overline{M}_{0,4}$, think of these as the boundary divisors