Math 476: Day 20

Ex: Let S^1 act on $(\mathbb{C}^n, \omega_{std})$ (where $\omega_{std} = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k = \sum r_k\, dr_k \wedge d\theta_k$) by

$$t \mapsto \psi_t = \text{mult. by } t$$

then the corresponding symplectic v.f. is

$$X^{\#} = \frac{\partial}{\partial \theta_1} + \dots + \frac{\partial}{\partial \theta_n}$$

& $\iota_{X^{\#}}\omega = -\sum r_k\, dr_k = d\left(-\frac{1}{2}\sum r_k^2\right)$, so the action is Hamiltonian w/ moment map

$$\mu : \mathbb{C}^n \to \mathfrak{u}(1)^* \cong \mathbb{R}$$
$$\vec{z} \mapsto -\frac{|\vec{z}|^2}{2} + \text{const.}$$

Image is a half-line

If we choose const $= 1/2$, then we see that $\mu^{-1}(0) = S^{2n-1}$.

Now, S^1 acts freely on S^{2n-1}, so the quotient is a mfld

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 \cong \mathbb{CP}^{n-1}$$ ← called the reduced space

(the projection $\begin{array}{c} S^1 \hookrightarrow S^{2n-1} \\ \downarrow \\ \mathbb{CP}^{n-1} \end{array}$ is an example of a Hopf map)

In general, w/ a Hamiltonian action we get a reduced space which is symplectic, the image of the moment map is convex & the pushforward measure on the image of the moment map is well-understood.

Thm (Marsden-Weinstein-Meyer): Suppose the cpt Lie group G acts in a Hamiltonian way on the symplectic mfd (M, ω), w/ moment map $\mu : M \to \mathfrak{g}^*$. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion & suppose G acts freely on $\mu^{-1}(0)$. Then

① The orbit space $M/\!\!/_0 G := \mu^{-1}(0)/G$ is a mfld

② $\pi : \mu^{-1}(0) \to M/\!\!/_0 G$ is a principal G-bundle (i.e. $\mu^{-1}(0)$ is locally a product)

③ There is a symplectic form ω_{red} on $M/\!\!/_0 G$ s.t. $i^*\omega = \pi^*\omega_{red}$

$$\begin{array}{ccccc} \mu^{-1}(0) & \xrightarrow{i} & M & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow{\scriptstyle \pi} & & & & \\ M/\!\!/_0 G & & & & \end{array}$$

Def: $(M/\!\!/_0 G, \omega_{red})$ is called the symplectic reduction or symplectic quotient or the Marsden-Weinstein-Meyer quotient of (M, ω) by G.

Notice that, by the equivariance of the moment map $\mu \circ \psi_g = Ad_g^* \circ \mu$, we see that, for any $p \in \mu^{-1}(0)$ & $g \in G$,

$$\mu(g \cdot p) = \mu(\psi_g(p)) = Ad_g^*(\mu(p)) = Ad_g^*(0) = 0,$$

so G does indeed act on $\mu^{-1}(0)$. Of course, in general this isn't true: consider the diagonal $SO(3)$ action on $(S^2)^n$.

We saw that the moment map $\mu : (S^2)^n \to so(3)^* \cong \mathbb{R}^3$ is given by $\mu(p_1, \dots, p_n) = p_1 + \dots + p_n$;

if $p_1 + \dots + p_n \neq 0$, then $\mu(g(p_1, \dots, p_n)) = \mu(g(p_1), \dots, g(p_n)) = g(p_1) + \dots + g(p_n) = g(p_1 + \dots + p_n)$

is not the same as $\mu(p_1, \dots, p_n)$ (though it does lie on the same sphere ... & note that the coadjoint orbits are symplectic)

so $SO(3)$ doesn't act on $\mu^{-1}(v)$ for $v \neq 0$ & so $\mu^{-1}(v)/SO(3)$ doesn't make sense.